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$\mathcal{C}^\infty$-Differentiable Spaces

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To Prof. Juan B. Sancho Guimerá

Preface

Differential geometry is traditionally regarded as the study of smooth manifolds, but sometimes this framework is too restrictive since it does not admit certain basic geometric intuitions. On the contrary, these geometric constructions are possible in the broader category of differentiable spaces. Let us indicate some natural objects which are differentiable spaces and not manifolds:

– Singular quadrics. Elementary surfaces of classical geometry such as a quadratic cone or a "doubly counted" plane are not smooth manifolds. Nevertheless, they have a natural differentiable structure, which is defined by means of the consideration of an appropriate algebra of differentiable functions.

For example, let us consider the quadratic cone X of equation $z^2-x^2-y^2=0$ in $\mathbb{R}^3$. It is a differentiable space whose algebra of differentiable functions is defined by

$$A := \mathcal{C}^\infty(\mathbb{R}^3)/\mathfrak{p}_X = \mathcal{C}^\infty(\mathbb{R}^3)/(z^2-x^2-y^2) \ ,$$

where $\mathfrak{p}_X$ stands for the ideal of $\mathcal{C}^\infty(\mathbb{R}^3)$ of all differentiable functions vanishing on X. In other words, differentiable functions on X are just restrictions of differentiable functions on $\mathbb{R}^3$.

Let us consider a more subtle example. Let Y be the plane in $\mathbb{R}^3$ of equation $z=0$. Of course this plane is a smooth submanifold. On the contrary, the "doubly counted" plane $z^2=0$ makes no sense in the language of smooth manifolds. It is another differentiable space with the same underlying topological space (the plane Y) but a different algebra of differentiable functions:

$$A := \mathcal{C}^\infty(\mathbb{R}^3)/(z^2) \ .$$

Note that A is not a subalgebra of $\mathcal{C}(Y,\mathbb{R})$, so that elements of A are not functions on Y in the set-theoretic sense.

More generally, any closed ideal $\mathfrak{a}$ of the Fréchet algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ defines a differentiable space (X,A), where

$$X := \{x \in \mathbb{R}^n : \ f(x)=0 \text{ for any } f \in \mathfrak{a}\}$$

is the underlying topological space and

$$A := \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$$

is the algebra of differentiable functions on this differentiable space. The pair (X,A) is the basic example of a differentiable subspace of $\mathbb{R}^n$ and the quotient map $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ is interpreted as the restriction morphism, i.e., for any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ the equivalence class $[f] \in \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ is said to be the restriction of f to the differentiable subspace under consideration.

– Fibres. Given a differentiable map $\varphi\colon \mathcal{V} \to \mathcal{W}$ between smooth manifolds, it may happen that some fibre $\varphi^{-1}(y)$ is not a smooth submanifold of $\mathcal{V}$, although it admits always a natural differentiable space structure.

– Intersections. Given a smooth manifold, the intersection of two smooth submanifolds may not be a smooth submanifold. On the contrary, intersections always exist in the category of differentiable spaces. For example, given two differentiable subspaces $(X, \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a})$ and $(Y, \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{b})$ of $\mathbb{R}^n$, where $\mathfrak{a}$ and $\mathfrak{b}$ are closed ideals of $\mathcal{C}^\infty(\mathbb{R}^n)$, the corresponding intersection is defined by the differentiable subspace

$$(X \cap Y \, , \, \mathcal{C}^\infty(\mathbb{R}^n)/\, \overline{\mathfrak{a} + \mathfrak{b}}\,) \, .$$

A more explicit example: The intersection of the parabola $y - x^2 = 0$ and the tangent $y = 0$ is the "doubly counted" origin

$$\big(\{(0,0)\}, \mathcal{C}^\infty(\mathbb{R}^2)/(y - x^2, y) = \mathbb{R}[x]/(x^2)\big)$$

and the number $2 = \dim \mathbb{R}[x]/(x^2)$ defines the *multiplicity* of the intersection.

More generally, *fibred products* exist in the category of differentiable spaces.

– Quotients. If we have a differentiable action of a Lie group G on a smooth manifold $\mathcal{V}$, it may occur that the topological quotient $\mathcal{V}/G$ admits no smooth manifold structure, even in such a simple case as a linear representation of a finite group. For example, if we consider the multiplicative action of $G = \{\pm 1\}$ on $\mathcal{V} = \mathbb{R}^3$, then the topological quotient $\mathcal{V}/G$ is not a topological manifold (nor a manifold with boundary). This example ruins any hope of a general result on the existence of quotients in the category of smooth manifolds under some reasonable hypotheses. On the contrary, in the category of differentiable spaces, we shall show the existence of quotients with respect to actions of compact Lie groups.

In particular, *orbifolds* usually have a natural structure of differentiable space.

– Infinitesimal neighbourhoods. The notion of an infinitesimal region naturally arises everywhere in differential geometry, but it is only used informally as a suggestive expression, due to the lack of a rigorous definition. Again, the language of differentiable spaces allows a suitable definition: Given a point x in a smooth manifold $\mathcal{V}$, the r-th infinitesimal neighbourhood of x is the differentiable subspace

$$U_x^r(\mathcal{V}) := \big(\{x\}, \mathcal{C}^\infty(\mathcal{V})/\mathfrak{m}_x^{r+1}\big) \ ,$$

where $\mathfrak{m}_x$ stands for the ideal of all differentiable functions vanishing at x.

The restriction of a differentiable function f to $U_x^r(\mathcal{V})$ is just the r-th jet of f at x, i.e., $j_x^r f = [f] \in \mathcal{C}^\infty(\mathcal{V})/\mathfrak{m}_x^{r+1}$. A systematic use of infinitesimal neighbourhoods simplifies and clarifies the theory of jets.

Let us show another application. The tangent bundle of an affine space $\mathbb{A}_n$ has a canonical trivialization $T\mathbb{A}_n = \mathbb{A}_n \times V_n$, where V_n is the vector space of free vectors on $\mathbb{A}_n$. Now let ∇ be a torsionless linear connection on a smooth manifold $\mathcal{V}$. For any point $x \in \mathcal{V}$, the restriction of the tangent bundle $T\mathcal{V}$ to $U_x^1(\mathcal{V})$ inherits a canonical trivialization (induced by parallel transport). In this sense, we may state that $(\mathcal{V}, \nabla)$ has the same infinitesimal structure as the affine space $\mathbb{A}_n$. This is a statement of the geometric meaning of a torsionless linear connection.

In a similar way, first infinitesimal neighbourhoods $U_x^1(\mathcal{V})$ in a Riemannian manifold $(\mathcal{V}, g)$ are always Euclidean, i.e., they have the same metric tangent structure as the first infinitesimal neighbourhood of any point in a Euclidean space. This statement captures the basic intuition about the notion of a Riemannian manifold as an infinitesimally Euclidean space.

With this, let us finish our list of examples of differentiable spaces. It should be sufficient to convince anybody of the necessity for an extension of the realm of differential geometry to include more general objects than smooth manifolds. A similar expansion occurred in the theory of algebraic varieties and analytic manifolds with the introduction of schemes and analytic spaces in the fifties. Following this path, Spallek introduced the category of differentiable spaces [59, 60, 62] which contains the category of smooth manifolds as a full subcategory. Moreover, all the foundational theorems on algebras of $\mathcal{C}^\infty$-differentiable functions are already at our disposal [26, 35, 71, 74]. In spite of this, the theory of general differentiable spaces has not been developed to the point of providing a handy tool in differential geometry. The aim of these notes is to develop the foundations of the theory of differentiable spaces in the best-behaved case: Spallek's ∞-standard differentiable spaces (henceforth simply differentiable spaces, since no other kind will be considered). These foundations will be developed so as to include the most basic tools at the same level as is standard in the theory of schemes and analytic spaces.

We would like to thank our friend R. Faro, who always has patience to attend to any question and to discuss it with us, and Prof. J. Muñoz Masqué, who taught us a course on rings of differentiable functions 25 years ago at the University of Salamanca.

We dedicate these notes to Professor Juan B. Sancho Guimerá, who directed around 1970 two doctoral dissertations [35, 37, 41] on the Localization theorem and always stressed to us its crucial importance in laying the foundations of differential geometry.

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Introduction

We shall develop the theory of differentiable spaces paralleling the theory of schemes introduced by Grothendieck [17] in algebraic geometry. First we choose the rings that should be considered as rings of differentiable functions, which are fixed to be quotients of $\mathcal{C}^\infty(\mathbb{R}^n)$ by some closed ideal $\mathfrak{a}$ (with respect to the Fréchet topology of uniform convergence on compact sets of functions and their derivatives). These Fréchet algebras $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ are named **differentiable algebras**[1], since $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ will be regarded as an algebra of differentiable functions on the closed subset $(\mathfrak{a})_0 := \{x \in \mathbb{R}^n : \ f(x) = 0 \ \forall f \in \mathfrak{a}\}$ of $\mathbb{R}^n$, even though this algebra may be full of nilpotent elements. Nevertheless, when $\mathfrak{a}$ is the ideal of all $\mathcal{C}^\infty$-functions vanishing on a given closed set $X \subseteq \mathbb{R}^n$, the quotient algebra $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ may be identified with a ring of real valued functions on X (in the usual set-theoretic sense), and in such a case we say that A is a **reduced** differentiable algebra.

Then we replace each differentiable algebra A by a ringed space (topological space with a sheaf of rings) $\mathrm{Spec}_r\, A$ called the **real spectrum** of A, since it is analogous to the prime spectrum used in algebraic geometry. Moreover, the analogue of a quasi-coherent sheaf on the prime spectrum is provided in our setting by the sheaf of modules defined by a Fréchet A-module. These ringed spaces $\mathrm{Spec}_r\, A$ define a category dual to the category of differentiable algebras, but it has the enormous advantage of leaving room for "recollement" procedures, as do any other kind of ringed spaces. Hence, a ringed space is said to be an **affine differentiable space** if it is isomorphic to the real spectrum of some differentiable algebra (the analogue of affine schemes), and **differentiable spaces** are defined to be ringed spaces where every point has an open neighbourhood which is an affine differentiable space (the analogue of schemes in algebraic geometry). Moreover, sheaves of Fréchet modules provide the analogue of quasi-coherent sheaves on schemes.

There has long been perceived the need for an extension of the framework of smooth manifolds in differential geometry, and there are several definitions that attempt to capture the intuitive concept of "non-smooth space with a differentiable structure". Over the years, some categories have appeared that are both large enough to include smooth manifolds and some other geometric objects, and small enough to admit a differential calculus. Therefore, the name *differentiable*

[1] Not to be confused with the notion of differentiable algebra as used in commutative algebra, which is simply an algebra with a derivation.

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space and similar terms have been used in a number of quite different senses. Let us discuss briefly some of them:

Spallek's differentiable spaces. Our differentiable spaces coincide with Spallek's differentiable spaces of a particular type (named ∞-standard); hence these notes fit naturally into the theory and applications of such spaces ([48] – [51] and [59] – [66]).

Synthetic differential geometry. At the end of the sixties Lawvere [24] proposed an axiomatic approach to the category of schemes, intended to be used in differential geometry and named "synthetic differential geometry". It was in this context that $\mathcal{C}^\infty$-schemes [9, 31] appeared, providing a model of this axiomatic theory. $\mathcal{C}^\infty$-rings are just $\mathbb{R}$-algebras where the composition $f(a_1, \ldots, a_n)$ with any smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ is defined, satisfying all the equations that hold between these functions; hence differentiable algebras are $\mathcal{C}^\infty$-rings. Then $\mathcal{C}^\infty$-ringed spaces are defined in the obvious way [9] and a $\mathcal{C}^\infty$-ringed space $\operatorname{Spec} A$ is associated to any $\mathcal{C}^\infty$-ring A (it coincides with our real spectrum whenever A is a differentiable algebra). Finally $\mathcal{C}^\infty$-schemes are introduced as $\mathcal{C}^\infty$-ringed spaces locally isomorphic to these local building blocks $\operatorname{Spec} A$. Therefore, differentiable spaces in our sense are $\mathcal{C}^\infty$-schemes; but $\mathcal{C}^\infty$-schemes lack a handy theory of sheaves of modules paralleling the useful theory of sheaves of Fréchet modules that we shall develop in the realm of differentiable spaces.

Real algebraic varieties of any kind are, of course, differentiable spaces. For example, up to isomorphisms, affine real algebraic varieties in the sense of Palais [44] are just pairs (X, A) where X is an algebraic set in $\mathbb{R}^n$ (closed subset defined by some polynomial equations) and A is the algebra of all polynomial functions $X \to \mathbb{R}$, so that A is just the quotient of $\mathbb{R}[x_1, \ldots, x_n]$ by the ideal of all polynomials vanishing on X. Hence any such algebraic variety inherits a natural structure of *reduced* affine differentiable space. In order to define non-affine real algebraic varieties, one introduces the sheaf $\mathcal{O}_X$ of regular functions on any algebraic set $X \subseteq \mathbb{R}^n$:

$$\mathcal{O}_X(U) \;:=\; \{p/q\colon\; p, q \in A,\; q(x) \neq 0 \;\forall x \in U\}\;.$$

These ringed spaces $(X, \mathcal{O}_X)$ are the local building blocks of the definition of real algebraic varieties given by Bochnak, Coste and Roy [2]. Again such algebraic varieties inherit a natural structure of *reduced* differentiable space. They consider only reduced algebraic varieties in spite of the critical role of non-reduced spaces in any infinitesimal consideration.

Orbifolds are defined to be locally isomorphic to $\mathbb{R}^n/G$, where G is some finite group [55, 69, 70]. Since such quotients of $\mathbb{R}^n$ are always differentiable spaces (see theorem 11.14) any differentiable orbifold is a differentiable space in our sense. We refer the reader to example 11.21 for a more detailed discussion of the relation between orbifolds and differentiable spaces.

Sikorski's differential spaces. According to Sikorski [57, 19] a differential space is a pair (X, A) where A is a set of continuous real functions on a topological space X such that:

1. X has the weakest topology such that all functions in A are continuous.
2. A is defined by local conditions: any function on X locally coinciding with functions in A belongs to A.
3. If $f_1, \dots, f_n \in A$ and $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$, then $\phi(f_1, \dots, f_n) \in A$.

Therefore, reduced affine differentiable spaces (in our sense) are differential spaces in this sense, and reduced differentiable spaces are differential spaces in the sense of Mostow [33, 19] which are the sheaf-theoretic version of Sikorski's differential spaces. But non-reduced differentiable spaces are never differential spaces in this sense. See [33] for a comparison of the notion of differential space in Mostow's sense with the definitions of Smith [58] and Chen [6, 7, 8].

Fröhlicher spaces. Fröhlicher and Kriegl [12, 23] defined a differential structure on a set X to be a family $\mathcal{C}$ of curves $\mathbb{R} \to X$ and a family $\mathcal{F}$ of functions $X \to \mathbb{R}$ such that

$$\mathcal{F} = \{f\colon X \to \mathbb{R} \mid f \circ c \in \mathcal{C}^\infty(\mathbb{R}),\ \forall c \in \mathcal{C}\}\ ,$$
$$\mathcal{C} = \{c\colon \mathbb{R} \to X \mid f \circ c \in \mathcal{C}^\infty(\mathbb{R}),\ \forall f \in \mathcal{F}\}\ .$$

Hence, by definition, differentiable functions on any $\mathcal{C}^\infty$-space of Fröhlicher and Kriegl are maps $X \to \mathbb{R}$ in the set-theoretic sense, so that non-reduced differentiable spaces are not $\mathcal{C}^\infty$-spaces in this sense. Moreover, such a simple reduced differentiable space as a convergent sequence with the limit point is not a $\mathcal{C}^\infty$-space of Fröhlicher.

Wiener's differential spaces. Our concept has nothing to do with the differential spaces introduced by Wiener [75].

Now let us briefly comment on the plan of these notes:

Chapter 1 presents the elementary theory of smooth manifolds in the spirit of differentiable spaces, so that their connections become clear. We try to take great care with the definitions, while omitting or at best just providing an indication of many proofs, since they are well-known.

Chapter 2 studies differentiable algebras $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$, where $\mathfrak{a}$ is a closed ideal in the usual Fréchet topology of $\mathcal{C}^\infty(\mathbb{R}^n)$. These algebras provide the building blocks for the construction of differentiable spaces since, by definition, the algebra of all differentiable functions on a certain open neighbourhood U of any point of a differentiable space is a differentiable algebra. Note that differentiable functions on U are not a certain kind of map $U \to \mathbb{R}$, since it may occur that $f^2 = 0$ while $f \neq 0$. Then, chapter 3 introduces differentiable spaces as ringed spaces locally modelled on differentiable algebras.

Chapter 4 is devoted to the study of some basic topological properties of differentiable spaces, including the existence of partitions of unity and the equivalence, in the affine case, between locally free sheaves of bounded rank and finitely generated projective modules over the ring of global differentiable functions.

Chapter 5 introduces differentiable subspaces and embeddings, with special attention to infinitesimal neighbourhoods. The main result is an embedding theorem for separated differentiable spaces whose topology has a countable basis.

Chapters 6 and 8 present an interlude of functional analysis, when the introduction of topological modules is unavoidable. Chapter 6 introduces locally convex modules over a Fréchet algebra A and studies topological tensor products $M \otimes_A N$ of these modules. Tensor products provide the basic tool for the theorem of existence of finite direct products and fibred products in the category of differentiable spaces, which is the main result of chapter 7. In particular, we have arbitrary finite intersections of differentiable subspaces and may define the fibre of any morphism of differentiable spaces over a subspace or a point.

In chapter 8 we study modules of fractions $S^{-1}M$ (see [35, 52]), including the Localization theorem for Fréchet modules. Modules of fractions are used in chapter 9 to study finite morphisms. The main result, analogous to Zariski's main theorem for algebraic varieties, states that a morphism of differentiable algebras $A \to B$ is finite if and only if $\operatorname{Spec}_r B \to \operatorname{Spec}_r A$ is a closed separated morphism with finite fibres of bounded degree, which is essentially a reformulation of Malgrange's preparation theorem [26].

In chapter 10 we use topological modules to introduce the module of relative differentials $\Omega_{B/A}$ for any morphism of differentiable algebras $A \to B$, and we study its properties, the main reference being [35]. These modules provide the basic tool for a differential calculus in the realm of differentiable spaces. We use them to define the sheaf of relative differentials $\Omega_{X/S}$ for any morphism of differentiable spaces $X \to S$, and to study smooth morphisms. The main result is the characterization, when the fibres are topological manifolds, of smooth morphisms over a reduced space as open maps with a locally free sheaf of relative differentials. In this chapter we also introduce formally smooth spaces and prove that a differentiable space X is formally smooth if and only if it is locally isomorphic to the Whitney space of a closed set in $\mathbb{R}^n$.

In the last chapter 11 we study quotients of smooth manifolds by compact Lie groups of transformations, which frequently are not smooth manifolds. We show that Schwarz's theorem [56] essentially states that such quotients have a structure of differentiable space, and study a natural stratification. We also briefly consider differentiable groups.

Finally we present two appendices. In the first one we study sheaves of Fréchet modules. For the sake of simplicity, in these notes we have deliberately avoided any topological structure on the sheaves that we introduce along the different chapters. Nevertheless, our sheaves typically have a natural topological structure, and the systematic use of sheaves of Fréchet modules greatly clarifies the theory, enabling us to restate some messy results with a more natural language and a more familiar aspect. In particular we define the inverse image of a sheaf of Fréchet modules.

In appendix B we introduce the space of r-jets of morphisms $X \to Y$, assuming that X is formally smooth. Since the r-jet of a morphism (or a function, a section, etc.) at a point x is just the restriction to the r-th infinitesimal neighbourhood of x, the theory of jets naturally involves non-reduced spaces. This appendix provides an example of the systematic use of the techniques developed in these notes, with a massive utilization of non-reduced differentiable spaces.

As to background, the following are prerequisite:

Commutative algebra: Tensor products, localization (modules of fractions), and completions [1].

Functional analysis: Fréchet spaces, Fréchet algebras, and topological tensor products [16, 18, 27, 29].

Sheaf theory: Basic operations with sheaves, and flabby sheaves [15].

Ideals of differentiable functions: Spectral synthesis of closed ideals , Whitney's ideals, Borel's theorem, Malgrange's preparation theorem, and Schwarz's theorem [26, 45, 71].

Differential geometry: Smooth manifolds, Lie groups, and actions of compact groups on manifolds [4, 73].

1 Differentiable Manifolds

The notion of smooth manifold, as well as those of analytic manifold and scheme, may be expressed appropriately in the language of ringed spaces. In this chapter we shall use this language, reformulating the traditional concepts of smooth manifold, differentiable map, submanifold, etc. Even in this limited context, the use of ringed spaces already presents some conceptual advantages. For example, the artificial concept of "maximal atlas" disappears in the definition of smooth manifold and no coordinate systems are required in the definition of differentiable map.

1.1 Smooth Manifolds

Definitions. Let $\mathcal{C}_X$ be the sheaf of real valued continuous functions on a topological space X. Subsheaves of $\mathbb{R}$-algebras of $\mathcal{C}_X$ are said to be **sheaves of continuous functions** on X. A **reduced ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of continuous functions on X (i.e., $\mathcal{O}_X(U)$ is a subalgebra of the algebra $\mathcal{C}(U, \mathbb{R})$ of all real valued continuous functions on U containing all constant functions, for any open set $U \subseteq X$).

Morphisms of reduced ringed spaces $\varphi\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ are defined to be continuous maps $\varphi\colon Y \to X$ such that $\varphi^* f := f \circ \varphi \in \mathcal{O}_Y(\varphi^{-1}U)$ whenever $f \in \mathcal{O}_X(U)$, so that φ induces a morphism of sheaves

$$\varphi^*\colon \mathcal{O}_X \longrightarrow \varphi_* \mathcal{O}_Y \ .$$

A morphism φ is said to be an **isomorphism** if there exists a morphism $\psi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $\varphi \circ \psi = Id$ and $\psi \circ \varphi = Id$; that is to say, if φ is a homeomorphism and $\varphi^*\colon \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ is an isomorphism of sheaves.

The following properties are obvious:

1. Let $(X, \mathcal{O}_X)$ be a reduced ringed space. If U is an open set in X, then $(U, \mathcal{O}_X|_U)$ is a reduced ringed space and the inclusion $U \hookrightarrow X$ is a morphism of reduced ringed spaces.
2. Compositions of morphisms of reduced ringed spaces also are morphisms of reduced ringed spaces.
3. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be reduced ringed spaces and let $\{U_i\}$ be an open cover of Y. A map $\varphi\colon Y \to X$ is a morphism of reduced ringed spaces if and only if so is the restriction $\varphi|_{U_i}\colon U_i \to X$ for any index i.

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Example 1.1. Any topological space X endowed with the sheaf $\mathcal{C}_X$ of real-valued continuous functions is a reduced ringed space. A map $\varphi\colon (Y,\mathcal{C}_Y) \to (X,\mathcal{C}_X)$ is a morphism if and only if it is continuous.

Example 1.2. $\mathcal{C}^\infty_{\mathbb{R}^n}$ will denote the sheaf of differentiable functions of class $\mathcal{C}^\infty$ on $\mathbb{R}^n$, so that $(\mathbb{R}^n,\mathcal{C}^\infty_{\mathbb{R}^n})$ is a reduced ringed space. It is easy to check that morphisms $(\mathbb{R}^m,\mathcal{C}^\infty_{\mathbb{R}^m}) \to (\mathbb{R}^n,\mathcal{C}^\infty_{\mathbb{R}^n})$ are just differentiable maps $\mathbb{R}^m \to \mathbb{R}^n$; i.e., maps $(f_1,\dots,f_n)\colon \mathbb{R}^m \to \mathbb{R}^n$ where $f_1,\dots,f_n \in \mathcal{C}^\infty(\mathbb{R}^m)$.

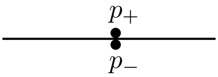
Definition. A reduced ringed space $(\mathcal{V},\mathcal{O}_\mathcal{V})$ is said to be a **smooth manifold** if every point of $\mathcal{V}$ admits an open neighbourhood in $\mathcal{V}$ which is isomorphic to an open subset of some $(\mathbb{R}^n,\mathcal{C}^\infty_{\mathbb{R}^n})$. In such a case we say that $\mathcal{O}_\mathcal{V}(U)$ is the ring of **differentiable functions** on the open subset $U \subseteq \mathcal{V}$ and we put

$$\mathcal{C}^\infty(U) \;:=\; \mathcal{O}_\mathcal{V}(U)\,.$$

A map $\varphi\colon \mathcal{V} \to \mathcal{W}$ between smooth manifolds is said to be **differentiable** when it is a morphism of reduced ringed spaces, and it is said to be a **diffeomorphism** when it is an isomorphism of reduced ringed spaces. The set of all differentiable maps $\mathcal{V} \to \mathcal{W}$ will be denoted by $\mathrm{Hom}(\mathcal{V},\mathcal{W})$.

If U is an open subset of a smooth manifold $\mathcal{V}$, note that $(U,\mathcal{O}_\mathcal{V}|_U)$ also is a smooth manifold and the inclusion $U \hookrightarrow \mathcal{V}$ is a differentiable map.

Remark 1.3. Smooth manifolds are not assumed to be separated (i.e. Hausdorff). A simple non-separated smooth manifold may be obtained by gluing two real lines U_+, U_- by means of the natural diffeomorphism $U_+ \setminus \{p_+\} \to U_- \setminus \{p_-\}$. The smooth manifold

$$\widetilde{\mathbb{R}} \;=\; U_+ \cup U_- \;=$$


so obtained is a real line with two non-separable points p_+, p_-. By definition, a continuous function on an open set $U \subseteq \widetilde{\mathbb{R}}$ is differentiable if and only if its restrictions to $U_+ \cap U$ and $U_- \cap U$ are of class $\mathcal{C}^\infty$ (in the usual sense). We use to say that $\widetilde{\mathbb{R}}$ is a real line where a point is split in two points.

Proposition 1.4. *Let $(\mathcal{V},\mathcal{O}_\mathcal{V})$ be a smooth manifold. A map*

$$\varphi = (f_1,\dots,f_n)\colon \mathcal{V} \longrightarrow \mathbb{R}^n$$

is differentiable if and only if $f_1,\dots,f_n \in \mathcal{O}_\mathcal{V}(\mathcal{V}) = \mathcal{C}^\infty(\mathcal{V})$. In particular, global differentiable functions $f \in \mathcal{C}^\infty(\mathcal{V})$ are just differentiable maps $f\colon \mathcal{V} \to \mathbb{R}$.

Proof. The problem being local, we may assume that $\mathcal{V}$ is diffeomorphic to an open subset U of $\mathbb{R}^m$. Now, if $f_1,\dots,f_n \in \mathcal{C}^\infty(U)$, then $F(f_1,\dots,f_n)$ is a differentiable function on $\varphi^{-1}(V)$ for any $F(x_1,\dots,x_n) \in \mathcal{C}^\infty(V)$ and any open set $V \subseteq \mathbb{R}^n$, so that $\varphi\colon U \to \mathbb{R}^n$ is a morphism of reduced ringed spaces.

Conversely, if φ is a morphism of reduced ringed spaces, then by definition $f_i = x_i \circ \varphi \in \mathcal{C}^\infty(U)$ for any $i = 1,\dots,n$.

□

Definition. Let U be an open subset of a smooth manifold $\mathcal{V}$. Some differentiable functions $u_1, \ldots, u_n \in \mathcal{C}^\infty(U)$ are said to define a **coordinate system** in U if the corresponding map $(u_1, \ldots, u_n)\colon U \to \mathbb{R}^n$ induces a diffeomorphism of U onto an open subset of $\mathbb{R}^n$. By definition, any point of a smooth manifold has a coordinate open neighbourhood.

Lemma 1.5. *Let K be a compact subset of a separated smooth manifold $\mathcal{V}$. If U is a neighbourhood of K, then there exists a differentiable function $h \in \mathcal{C}^\infty(\mathcal{V})$ such that*

1. $h = 1$ *on an open neighbourhood of K.*
2. $U \supseteq \mathrm{Supp}\, h := \overline{\{x \in \mathcal{V}\colon\ h(x) \neq 0\}}$.
3. $0 \leq h(x) \leq 1$ *at any point $x \in X$.*

Proof. Standard.

□

Corollary 1.6. *Let f be a differentiable function on an open subset U of a separated smooth manifold $\mathcal{V}$. If $p \in U$, then f coincides on a neighbourhood of p with some differentiable function $F \in \mathcal{C}^\infty(\mathcal{V})$.*

Hence we have natural isomorphisms

$$\mathcal{C}^\infty(\mathcal{V})/\mathfrak{n}_p \;=\; \mathcal{O}_p \quad , \quad \mathcal{C}^\infty(\mathcal{V})_p \;=\; \mathcal{O}_p \;,$$

where $\mathcal{O}_p$ is the ring of germs at p of differentiable functions, $\mathfrak{n}_p$ is the ideal of all global differentiable functions vanishing at some neighbourhood of p, and $\mathcal{C}^\infty(\mathcal{V})_p = S^{-1}\mathcal{C}^\infty(\mathcal{V})$ denotes the localization (ring of fractions) of $\mathcal{C}^\infty(\mathcal{V})$ with respect to the multiplicative system $S = \{s \in \mathcal{C}^\infty(\mathcal{V})\colon\ s(p) \neq 0\}$.

Proof. Let K be a compact neighbourhood of p and let $h \in \mathcal{C}^\infty(\mathcal{V})$ be the function of 1.5. Then $F = fh$, extended by zero, is the required global differentiable function.

Therefore, the morphisms $\mathcal{C}^\infty(\mathcal{V})/\mathfrak{n}_p \to \mathcal{O}_p$, $[f] \mapsto f_p$, and $\mathcal{C}^\infty(\mathcal{V})_p \to \mathcal{O}_p$, $[f/s] \mapsto f_p/s_p$, are surjective (where the germ at p of any function h is denoted by h_p). The first morphism is clearly injective. The second one also is injective: If $f_p/s_p = 0$ in $\mathcal{O}_p$, then f vanishes on some neighbourhood U of p and, by 1.5, there exists some $h \in \mathcal{C}^\infty(\mathcal{V})$ such that $\mathrm{Supp}\, h \subseteq U$ and $h(p) = 1$ (hence $h \in S$). Since $hf = 0$ we conclude that $[f/s] = [fh/sh] = 0$ in $\mathcal{C}^\infty(\mathcal{V})_p$.

□

Definition. A **partition of unity** subordinated to an open cover $\{U_i\}_{i \in I}$ of a smooth manifold $\mathcal{V}$ is a family of global differentiable functions $\{\phi_i\}_{i \in I}$ such that

1. $\phi_i \geq 0$ and $\mathrm{Supp}\, \phi_i \subseteq U_i$ for any index $i \in I$.
2. $\{\mathrm{Supp}\, \phi_i\}_{i \in I}$ is a locally finite family.
3. $\sum_i \phi_i = 1$ (the sum is full-sense by 2).

Theorem of existence of partitions of unity. *Let $\mathcal{V}$ be a separated smooth manifold whose topology has a countable basis. Any open cover $\{U_i\}_{i\in I}$ of $\mathcal{V}$ has a subordinated partition of unity.*

Proof. Standard. □

Corollary 1.7. *Let X, Y be disjoint closed subsets of a separated smooth manifold $\mathcal{V}$ whose topology has a countable basis. There exists a global differentiable function $0 \leq f \leq 1$ such that $f = 1$ on a neighbourhood of X and $f = 0$ on a neighbourhood of Y.*

1.2 Taylor Expansions

Let p be a point of a smooth manifold $\mathcal{V}$ and let $\mathcal{O}_p$ be the ring of germs at p of differentiable functions (it is the stalk at p of the structural sheaf $\mathcal{O}_{\mathcal{V}} = \mathcal{C}^\infty_{\mathcal{V}}$).

Theorem 1.8. *The ring of germs $\mathcal{O}_p$ has a unique maximal ideal*

$$\mathfrak{m}_p \;=\; \{f \in \mathcal{O}_p \colon\; f(p) = 0\}\ .$$

If $(u_1, \ldots, u_n)$ is a coordinate system at p and $p_i = u_i(p)$, then $\mathfrak{m}_p$ is generated by the germs at p of the functions $u_1 - p_1, \ldots, u_n - p_n$.

Proof. The morphism of rings $\mathcal{O}_p \to \mathbb{R}$, $f \mapsto f(p)$, is surjective, hence its kernel $\mathfrak{m}_p$ is a maximal ideal. It is the unique maximal ideal of $\mathcal{O}_p$ because any germ $f \in \mathcal{O}_p$, such that $f(p) \neq 0$, is invertible in $\mathcal{O}_p$.

For the second part, the problem being local, we may replace $\mathcal{V}$ by $\mathbb{R}^n$ and $u_1, \ldots, u_n$ by the cartesian coordinates $x_1, \ldots, x_n$ of $\mathbb{R}^n$, so that $p = (p_1, \ldots, p_n)$. By 1.6, any germ $f \in \mathfrak{m}_p$ is represented by a differentiable function defined on $\mathbb{R}^n$; therefore, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we may consider the function

$$g(t) \;=\; f(p + t(x - p)) \quad , \quad t \in [0, 1]\ .$$

Since $f(p) = 0$, we have

$$\begin{aligned} f(x) = f(x) - f(p) = g(1) - g(0) = \int_0^1 g'(t)\,\mathrm{d}t \\ = \sum_{i=1}^n \int_0^1 (x_i - p_i)\frac{\partial f}{\partial x_i}(p + t(x-p))\,\mathrm{d}t \\ = \sum_{i=1}^n (x_i - p_i)\int_0^1 \frac{\partial f}{\partial x_i}(p + t(x-p))\,\mathrm{d}t = \sum_{i=1}^n (x_i - p_i)h_i(x) \end{aligned}$$

and, taking germs at p, we conclude that $f \in (x_1 - p_1, \ldots, x_n - p_n)$.

Finally, the inclusion $(x_1 - p_1, \ldots, x_n - p_n) \subseteq \mathfrak{m}_p$ is obvious. □

The above theorem provides the Taylor expansion of a differentiable function at a point of coordinates $(p_1, \dots, p_n)$. Let us denote by $O(r)$ any element of the ideal $\mathfrak{m}_p^r$. The theorem shows that the ideal $\mathfrak{m}_p^r$ is generated by the products

$$(u_1 - p_1)^{i_1} \dots (u_n - p_n)^{i_n} \ ,$$

where $i_1 + \dots + i_n = r$. Since any germ $f \in \mathcal{O}_p$ is $f = a + O(1)$ for some $a \in \mathbb{R}$, it follows that

$$O(r) = \sum_{i_1+\dots+i_n=r} a_{i_1\dots i_n}(u_1 - p_1)^{i_1} \dots (u_n - p_n)^{i_n} + O(r+1)$$

and we obtain the Taylor series of any germ f at p:

$$\begin{aligned} f &= a + O(1) = a + \sum_i a_i(u_i - p_i) + O(2) \\ &= a + \sum_i a_i(u_i - p_i) + \sum_{i \le j} a_{ij}(u_i - p_i)(u_j - p_j) + O(3) \\ &= \sum_{i_1+\dots+i_n \le r} a_{i_1\dots i_n}(u_1 - p_1)^{i_1} \dots (u_n - p_n)^{i_n} + O(r+1) = \dots \end{aligned}$$

Of course, by successive partial derivations we get the standard result

$$a_{i_1,\dots,i_n} = \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1+\dots+i_n} f}{\partial u_1^{i_1} \cdots \partial u_n^{i_n}}(p) \ .$$

Let us denote by $j_p^r f$ the Taylor expansion of order r (or r-jet) of f at p. The above computation gives directly the following

Corollary 1.9. *$\mathfrak{m}_p^{r+1}$ is the ideal of all germs $f \in \mathcal{O}_p$ such that $j_p^r f = 0$. Moreover, we have an isomorphism*

$$\begin{aligned} \mathcal{O}_p/\mathfrak{m}_p^{r+1} &= \mathbb{R}[u_1 - p_1, \dots, u_n - p_n]/(u_1 - p_1, \dots, u_n - p_n)^{r+1} \\ [f] &\mapsto j_p^r f \end{aligned}$$

Remark 1.10. Using the above isomorphism, we may compute the $\mathfrak{m}_p$-adic completion of $\mathcal{O}_p$,

$$\widehat{\mathcal{O}}_p := \varprojlim_r \mathcal{O}_p/\mathfrak{m}_p^{r+1} = \mathbb{R}[[u_1 - p_1, \dots, u_n - p_n]] \ ,$$

i.e., $\widehat{\mathcal{O}}_p$ is the $\mathbb{R}$-algebra of formal power series in the variables $u_1 - p_1, \dots, u_n - p_n$. Note that the natural morphism

$$j_p \colon \mathcal{O}_p \longrightarrow \widehat{\mathcal{O}}_p = \mathbb{R}[[u_1 - p_1, \dots, u_n - p_n]] \quad , \quad f \mapsto \varprojlim [f]_r = j_p f$$

maps any germ f into the Taylor expansion $j_p f$ of f at p.

Corollary 1.11. *Let $\mathcal{V}$ be a separated smooth manifold and let $\mathfrak{m}_{\mathcal{V},p}$ be the ideal of all differentiable functions $f \in \mathcal{C}^\infty(\mathcal{V})$ vanishing at a given point $p \in \mathcal{V}$. For any $r \geq 0$, we have*

$$\mathfrak{m}_{\mathcal{V},p}^{r+1} \;=\; \{f \in \mathcal{C}^\infty(\mathcal{V}) : \; j_p^r f = 0\}$$

and a natural isomorphism

$$\mathcal{C}^\infty(\mathcal{V})/\mathfrak{m}_{\mathcal{V},p}^{r+1} \;=\; \mathcal{O}_p/\mathfrak{m}_p^{r+1} \; .$$

Proof. Let $\mathfrak{n}_p$ be the ideal of all differentiable functions on $\mathcal{V}$ vanishing on some neighbourhood of p. Let us show that $\mathfrak{n}_p^2 = \mathfrak{n}_p$, hence $\mathfrak{n}_p^r = \mathfrak{n}_p$ for any $r > 0$. Given a function $f \in \mathfrak{n}_p$ vanishing on some neighbourhood U of p, let us consider $h \in \mathcal{C}^\infty(\mathcal{V})$ such that $\operatorname{Supp} h \subseteq U$ and $h = 1$ on a neighbourhood of p. Then $fh = 0$ on $\mathcal{V}$, hence $f = f(1-h) \in \mathfrak{n}_p^2$ and we conclude that $\mathfrak{n}_p = \mathfrak{n}_p^2$.

Now, it is obvious that $\mathfrak{n}_p \subseteq \mathfrak{m}_{\mathcal{V},p}$, hence $\mathfrak{n}_p = \mathfrak{n}_p^{r+1} \subseteq \mathfrak{m}_{\mathcal{V},p}^{r+1}$ for any $r > 0$. Since $\mathcal{C}^\infty(\mathcal{V})/\mathfrak{n}_p = \mathcal{O}_p$ (by 1.6), we conclude that $\mathcal{C}^\infty(\mathcal{V})/\mathfrak{m}_{\mathcal{V},p}^{r+1} = \mathcal{O}_p/\mathfrak{m}_p^{r+1}$.

Finally, considering the kernel of the morphism

$$\mathcal{C}^\infty(\mathcal{V}) \longrightarrow \mathcal{C}^\infty(\mathcal{V})/\mathfrak{m}_{\mathcal{V},p}^{r+1} \;=\; \mathcal{O}_p/\mathfrak{m}_p^{r+1}, \qquad f \mapsto [f] = j_p^r f$$

we obtain that $\mathfrak{m}_{\mathcal{V},p}^{r+1} = \{f \in \mathcal{C}^\infty(\mathcal{V}) : j_p^r f = 0\}$.

□

Remark 1.12. Using the above isomorphism, we may obtain the $\mathfrak{m}_{\mathcal{V},p}$-adic completion of $\mathcal{C}^\infty(\mathcal{V})$,

$$\mathcal{C}^\infty(\mathcal{V})\widehat{\;}_p := \varprojlim_r \mathcal{C}^\infty(\mathcal{V})/\mathfrak{m}_{\mathcal{V},p}^{r+1} = \varprojlim_r \mathcal{O}_p/\mathfrak{m}_p^{r+1} = \widehat{\mathcal{O}}_p = \mathbb{R}[[u_1 - p_1, \ldots, u_n - p_n]].$$

Again, the natural morphism

$$j_p \colon \mathcal{C}^\infty(\mathcal{V}) \longrightarrow \mathcal{C}^\infty(\mathcal{V})\widehat{\;}_p \;=\; \mathbb{R}[[u_1 - p_1, \ldots, u_n - p_n]] \quad , \quad f \mapsto \varprojlim [f]_r = j_p f$$

maps any differentiable function f into the Taylor expansion $j_p f$ of f at p.

1.3 Tangent Space

Let p be a point of a smooth manifold $\mathcal{V}$ and let $\mathcal{O}_p$ be the ring of germs at p of differentiable functions.

Definition. A **tangent vector** at a point $p \in \mathcal{V}$ is defined to be a derivation $D \colon \mathcal{O}_p \to \mathcal{O}_p/\mathfrak{m}_p = \mathbb{R}$. That is to say, it is an $\mathbb{R}$-linear map $D \colon \mathcal{O}_p \to \mathbb{R}$ such that

$$D(fg) \;=\; (Df)g(p) + f(p)(Dg) \; .$$

The real vector space $T_p\mathcal{V} := \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_p, \mathbb{R})$ of all tangent vectors at p is said to be the **tangent space** to $\mathcal{V}$ at p. The dimension of $\mathcal{V}$ at p is defined to be the dimension of $T_p\mathcal{V}$.

Note that if U is an open neighbourhood of p, then $T_pU = T_p\mathcal{V}$ because $\mathcal{O}_{U,p} = \mathcal{O}_{\mathcal{V},p}$.

Given a coordinate system $(u_1, \ldots, u_n)$ at p, we denote by $(\partial/\partial u_i)_p$ the tangent vector

$$\left(\frac{\partial}{\partial u_i}\right)_p f := \frac{\partial f}{\partial u_i}(p) \ .$$

Proposition 1.13. *If $(u_1, \ldots, u_n)$ is a local coordinate system at a point $p \in \mathcal{V}$, then $\{(\partial/\partial u_1)_p, \ldots, (\partial/\partial u_n)_p\}$ is a basis of $T_p\mathcal{V}$.*

Proof. If $D \in T_p\mathcal{V}$ and $Du_i = 0$ for any $1 \leq i \leq n$, then $D = 0$. In fact, if $f \in \mathcal{O}_p$, by 1.8 we have

$$f - f(p) = \sum_{i=1}^{n} h_i \cdot (u_i - p_i) \ ,$$

$$Df = \sum_{i=1}^{n} (Dh_i)(u_i(p) - p_i) + \sum_{i=1}^{n} h_i(p)(Du_i) = 0 + 0 \ .$$

In particular, we obtain the following formula for any tangent vector D:

$$D \ = \ \sum_{i=1}^{n} (Du_i) \left(\frac{\partial}{\partial u_i}\right)_p \tag{1.13.1}$$

since both derivations clearly coincide on the coordinates $u_1, \ldots, u_n$.

Finally we show that the derivations $(\partial/\partial u_1)_p, \ldots, (\partial/\partial u_n)_p$ are linearly independent. If $D = \sum_i \lambda_i (\partial/\partial u_i)_p = 0$, it is immediate that $\lambda_i = Du_i = 0$.

□

Definition. Let $\varphi\colon \mathcal{V} \to \mathcal{W}$ be a differentiable map between smooth manifolds, let $p \in \mathcal{V}$ and let $q = \varphi(p)$. The **tangent linear map** $\varphi_*\colon T_p\mathcal{V} \to T_q\mathcal{W}$ is defined to be

$$(\varphi_* D)f \ = \ D(f \circ \varphi) \ .$$

Once we take a coordinate neighbourhood $(V; v_1, \ldots, v_m)$ of q and a coordinate neighbourhood $(U; u_1, \ldots, u_n)$ of p such that $\varphi(U) \subseteq V$, then the map $\varphi\colon U \to V$ is defined by some equations

$$v_i = f_i(u_1, \ldots, u_n) \quad , \quad 1 \leq i \leq m \ ,$$

where $f_i := \varphi^* v_i = v_i \circ \varphi \in \mathcal{C}^\infty(U)$. By 1.13.1 we have

$$\varphi_* \left(\frac{\partial}{\partial u_j}\right)_p \ = \ \sum_{i=1}^{m} \frac{\partial f_i}{\partial u_j}(p) \left(\frac{\partial}{\partial v_i}\right)_p \ ,$$

so that the matrix of φ_* with respect to the bases $\{(\partial/\partial u_j)_p\}$ and $\{(\partial/\partial v_i)_q\}$ is just the jacobian matrix at p

$$\left(\frac{\partial f_i}{\partial u_j}(p)\right)$$

and the Inverse function theorem may be restated as follows:

Inverse function theorem. *Let $\varphi\colon \mathcal{V} \to \mathcal{W}$ be a differentiable map between smooth manifolds. If the tangent linear map $\varphi_*\colon T_p\mathcal{V} \to T_{\varphi p}\mathcal{W}$ is an isomorphism, then φ is a local diffeomorphism at p.*

Definition. The dual vector space $T_p^*\mathcal{V}$ of the tangent space $T_p\mathcal{V}$ is said to be the **cotangent space** of $\mathcal{V}$ at p. Its elements are said to be 1-forms at p. The **differential** $\mathrm{d}_p f$ of a germ $f \in \mathcal{O}_p$ is defined to be the 1-form

$$(\mathrm{d}_p f)(D) := Df \quad , \quad D \in T_p\mathcal{V} .$$

It is easy to check that the map $\mathrm{d}_p\colon \mathcal{O}_p \to T_p^*\mathcal{V}$, $f \mapsto \mathrm{d}_p f$, is a derivation:

1. $\mathrm{d}_p(f+g) = \mathrm{d}_p f + \mathrm{d}_p g$.
2. $\mathrm{d}_p(fg) = g(p)(\mathrm{d}_p f) + f(p)(\mathrm{d}_p g)$.
3. $\mathrm{d}_p\lambda = 0$ for any $\lambda \in \mathbb{R}$ (hence d_p is $\mathbb{R}$-linear).

If $(U; u_1, \ldots, u_n)$ is a coordinate neighbourhood of p, then the dual basis of $\{(\partial/\partial u_1)_p, \ldots, (\partial/\partial u_n)_p\}$ is just $\{\mathrm{d}_p u_1, \ldots, \mathrm{d}_p u_n\}$. Hence $\{\mathrm{d}_p u_1, \ldots, \mathrm{d}_p u_n\}$ is a basis of the cotangent space $T_p^*\mathcal{V}$, and we have

$$\mathrm{d}_p f = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}(p)\, \mathrm{d}_p u_i ,$$

because both 1-forms coincide on the basis $\{(\partial/\partial u_1)_p, \ldots, (\partial/\partial u_n)_p\}$. Moreover, if $\varphi\colon \mathcal{V} \to \mathcal{W}$ is a differentiable map and $\varphi_*\colon T_p\mathcal{V} \to T_q\mathcal{W}$ is the tangent linear map at p, then $\varphi^*\colon T_q^*\mathcal{W} \to T_p^*\mathcal{V}$ will denote the dual linear map; that is to say, $(\varphi^*\omega)(D) := \omega(\varphi_* D)$. It is easy to check that $\varphi^*(\mathrm{d}_q f) = \mathrm{d}_p(\varphi^* f)$ for any germ $f \in \mathcal{O}_q$.

Proposition 1.14. *There exists a canonical $\mathbb{R}$-linear isomorphism*

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = T_p^*\mathcal{V} \qquad , \qquad [f] \mapsto \mathrm{d}_p f .$$

Proof. It is easy to check that $D(\mathfrak{m}_p^2) = 0$ for any derivation $D \in T_p\mathcal{V}$. Therefore

$$T_p\mathcal{V} = \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_p, \mathbb{R}) = \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_p/\mathfrak{m}_p^2, \mathbb{R}).$$

Since $\mathcal{O}_p/\mathfrak{m}_p^2 = \mathbb{R} \oplus \mathfrak{m}_p/\mathfrak{m}_p^2$, we have $\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_p/\mathfrak{m}_p^2, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R})$ and then $T_p\mathcal{V} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R})$. This duality is clearly defined by the pairing

$$T_p\mathcal{V} \times \mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow \mathbb{R} \quad , \quad (D, [f]) \mapsto Df = \mathrm{d}_p f(D) .$$

□

Note that the differential of a function f at p coincides, via the above isomorphism, with the residue class of the increment of f at p,

$$[f - f(p)] \;=\; \mathrm{d}_p f \;.$$

Proposition 1.15. *Let $f_1, \ldots, f_n$ be differentiable functions on a smooth manifold $\mathcal{V}$ and let $p \in \mathcal{V}$. If $\{\mathrm{d}_p f_1, \ldots, \mathrm{d}_p f_n\}$ is a basis of $T_p^*\mathcal{V}$, then $(f_1, \ldots, f_n)$ is a local coordinate system at p.*

Proof. Let us consider the differentiable map $\varphi = (f_1, \ldots, f_n)\colon \mathcal{V} \to \mathbb{R}^n$ and let $q = \varphi(p)$. We have $\varphi^*(\mathrm{d}_q x_i) = \mathrm{d}_p(\varphi^* x_i) = \mathrm{d}_p f_i$, so that $\varphi^*\colon T_q^*\mathbb{R}^n \to T_p^*\mathcal{V}$ is an isomorphism. Therefore $\varphi_*\colon T_p\mathcal{V} \to T_q\mathbb{R}^n$ is an isomorphism and the Inverse function theorem let us conclude that φ induces a diffeomorphism of an open neighbourhood of p onto an open subset of $\mathbb{R}^n$.

□

1.4 Smooth Submanifolds

Definition. Let $(X, \mathcal{O}_X)$ be a reduced ringed space and let Y be a subspace of X. The functions in $\mathcal{O}_X$, when restricted to Y, define a sheaf of continuous functions $\mathcal{O}_Y$ on Y, called the **induced sheaf**. By definition, if f is a continuous real-valued function on an open set $V \subseteq Y$, then $f \in \mathcal{O}_Y(V)$ just when every point $y \in Y$ has an open neighbourhood U in X such that $f|_{U\cap V} = F|_{U\cap V}$ for some $F \in \mathcal{O}_X(U)$. Therefore $(Y, \mathcal{O}_Y)$ is a reduced ringed space and the inclusion map $i\colon (Y, \mathcal{O}_Y) \hookrightarrow (X, \mathcal{O}_X)$ is a morphism of reduced ringed spaces. Moreover, if $\varphi\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a morphism of reduced ringed spaces and $\varphi(Z) \subseteq Y$, then $\varphi\colon (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ is a morphism of reduced ringed spaces.

Definition. Let Y be a subspace of a smooth manifold $(\mathcal{V}, \mathcal{O}_\mathcal{V})$. If Y, endowed with the induced sheaf $\mathcal{O}_Y$, is a smooth manifold, then we say that Y is a **smooth submanifold** of $\mathcal{V}$.

For example, if we consider the smooth manifold $(\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}, \mathcal{C}^\infty_{\mathbb{R}^n})$, then the induced sheaf on $\mathbb{R}^m \times \{0\}$ is just $\mathcal{C}^\infty_{\mathbb{R}^m}$; hence $\mathbb{R}^m \times 0$ is a smooth submanifold of $\mathbb{R}^n$.

Lemma 1.16. *Let Y be a smooth submanifold of a smooth manifold $\mathcal{V}$ and let $i\colon Y \hookrightarrow \mathcal{V}$ be the inclusion map. The linear tangent map $i_*\colon T_yY \to T_y\mathcal{V}$ is injective at any point $y \in Y$.*

Proof. The restriction morphism $i^*\colon \mathcal{O}_{\mathcal{V},y} \to \mathcal{O}_{Y,y}$ is surjective by definition of the induced sheaf $\mathcal{O}_Y$. Now, if $i_*D = 0$, then

$$0 = (i_*D)(\mathcal{O}_{\mathcal{V},y}) = D(i^*\mathcal{O}_{\mathcal{V},y}) = D(\mathcal{O}_{Y,y})$$

and we conclude that $D = 0$.

□

Theorem 1.17. *Let $f_1, \dots, f_r$ be differentiable functions on a smooth manifold $\mathcal{V}$ and let $Y = \{x \in \mathcal{V}\colon\ f_1(x) = 0, \dots, f_r(x) = 0\}$. If $\mathrm{d}_y f_1, \dots, \mathrm{d}_y f_r$ are linearly independent at any point $y \in Y$, then Y is a smooth submanifold of codimension r and T_yY coincides, via i_*, with the incident of the linear span of $\mathrm{d}_y f_1, \dots, \mathrm{d}_y f_r$.*

Proof. Let $\{\mathrm{d}_y f_1, \dots, \mathrm{d}_y f_r, \mathrm{d}_y f_{r+1}, \dots, \mathrm{d}_y f_n\}$ be a basis of $T_y^*\mathcal{V}$.

By 1.15, $(f_1, \dots, f_n)$ is a coordinate system in an open neighbourhood U of y, so that $\varphi = (f_1, \dots, f_n)\colon U \to \mathbb{R}^n$ induces a diffeomorphism of U onto an open subset $U' \subseteq \mathbb{R}^n$. Hence φ defines an isomorphism of reduced ringed spaces $\varphi\colon Y \cap U \to \varphi(Y \cap U)$. Now, $\varphi(Y \cap U) = (0 \times \mathbb{R}^{n-r}) \cap U'$ is an open subset of $0 \times \mathbb{R}^{n-r} = \mathbb{R}^{n-r}$, and we obtain that any point $y \in Y$ has an open neighbourhood $Y \cap U$ which is isomorphic to an open subset of $\mathbb{R}^{n-r}$. Since $\mathcal{C}^\infty_{\mathbb{R}^{n-r}}$ is the sheaf induced by $\mathcal{C}^\infty_{\mathbb{R}^n}$ on $0 \times \mathbb{R}^{n-r}$, we have that $(Y, \mathcal{O}_Y)$ is a smooth submanifold of codimension r.

Finally, if $D \in T_yY$, then

$$(\mathrm{d}_y f_k)(i_*D) = (i_*D)f_k = D(f_k \circ i) = D(0) = 0 \quad , \quad 1 \le k \le r \ ,$$

so that i_*T_yY is contained in the incident of $< \mathrm{d}_y f_1, \dots, \mathrm{d}_y f_r >$. Both linear subspaces coincide because they have the same dimension ($n - r = \dim(i_*T_yY)$ by 1.16).

□

Theorem 1.18. *Let Y be a smooth submanifold of codimension r of a smooth manifold $\mathcal{V}$. Any point p of Y has some coordinate neighbourhood $(U; u_1, \dots, u_n)$ in $\mathcal{V}$ such that*

$$Y \cap U \ = \ \{x \in U\colon\ u_1(x) = 0, \dots, u_r(x) = 0\} \ .$$

In particular, Y is a locally closed subspace of $\mathcal{V}$.

Proof. We may assume that $\mathcal{V} = \mathbb{R}^n$. By 1.16, the linear map $i^*\colon T_p^*\mathbb{R}^n \to T_p^*Y$ is surjective, hence $i^*(\mathrm{d}_p x_1), \dots, i^*(\mathrm{d}_p x_n)$ span T_p^*Y and we may assume that $\{\mathrm{d}_p \bar{x}_1, \dots, \mathrm{d}_p \bar{x}_m\}$ is a basis of T_p^*Y, where $\bar{x}_k = i^*x_k = x_k|_Y$ and $m = n - r$. By 1.15, $(\bar{x}_1, \dots, \bar{x}_m)$ is a coordinate system in some neighbourhood V of p in Y, so that it defines a diffeomorphism of V onto an open subset $V' \subseteq \mathbb{R}^m$. In V we have

$$\bar{x}_{m+j} \ = \ f_j(\bar{x}_1, \dots, \bar{x}_m) \quad , \quad f_j \in \mathcal{C}^\infty(V') \ .$$

Now, since Y is a topological subspace of $\mathbb{R}^n$, there exists an open subset $U \subseteq \mathbb{R}^n$ such that $V = Y \cap U$ and, replacing U by $U \cap (V' \times \mathbb{R}^r)$, we may assume that V' is the image of U by the projection $(x_1, \dots, x_m)\colon \mathbb{R}^n \to \mathbb{R}^m$. Let us consider the following differentiable functions $u_1, \dots, u_n \in \mathcal{C}^\infty(U)$:

$$\begin{aligned} u_i &= x_i & , \quad & 1 \le i \le m \ , \\ u_{m+j} &= x_{m+j} - f_j(x_1, \dots, x_m) & , \quad & 1 \le j \le r \ . \end{aligned}$$

It is clear that $\mathrm{d}_p u_1, \dots, \mathrm{d}_p u_n$ are linearly independent; hence, by 1.15, we may assume that $(u_1, \dots, u_n)$ is a coordinate system in U. Finally it is easy to check that

$$Y \cap U = \{x \in U\colon\ u_{m+1}(x) = 0, \ldots, u_{m+r}(x) = 0\}\ .$$

□

Locally closed subspaces: A subspace Y of a topological space X is said to be **locally closed** if every point $y \in Y$ has an open neighbourhood U_y in X such that $U_y \cap Y$ is closed in U_y.

Any locally closed subspace of X is a closed set of an open subset of X.
In fact, Y is closed in the open set $U = \bigcup_y U_y$ because so it is in the open cover $\{U_y\}$ of U.

The concept of locally closed subspace is local in Y: *If each point of Y has an open neighbourhood in Y which is locally closed in X, then Y is locally closed in X.*

In fact, given $y \in Y$, let V be an open neighbourhood of y in Y which is locally closed in X and let U be an open set in X such that $V = U \cap Y$. By definition y has an open neighbourhood U_y in X such that $U_y \cap V$ is closed in U_y. Then $U'_y = U_y \cap U$ is an open neighbourhood of y in X and $U'_y \cap Y = U_y \cap U \cap Y = U_y \cap V$ is closed in U_y; hence it is closed in $U'_y \subseteq U_y$.

Theorem 1.19. *Let $\varphi\colon \mathcal{V} \to \mathcal{W}$ be a differentiable map between smooth manifolds, let $p \in \mathcal{V}$ and let $q = \varphi(p)$. The following conditions are equivalent:*

1. *The tangent linear map $\varphi_*\colon T_p\mathcal{V} \to T_q\mathcal{W}$ is injective.*
2. *There exists an open neighbourhood V of p in $\mathcal{V}$ such that φ induces a diffeomorphism of V onto a smooth submanifold of $\mathcal{W}$.*

Proof. Standard.

□

Definition. A differentiable map $\varphi\colon \mathcal{V} \to \mathcal{W}$ between smooth manifolds is said to be a **local embedding** at a point $p \in \mathcal{V}$ when it satisfies the above equivalent conditions.

Definition. A differentiable map $\mathcal{V} \to \mathcal{W}$ between smooth manifolds is said to be an **embedding** (resp. **closed embedding**) if it induces a diffeomorphism of $\mathcal{V}$ onto a smooth submanifold (resp. closed smooth submanifold) of $\mathcal{W}$.

Theorem 1.20. *Let $\varphi\colon \mathcal{V} \to \mathcal{W}$ be a differentiable map between smooth manifolds. The following conditions are equivalent:*

1. *$\varphi\colon \mathcal{V} \to \mathcal{W}$ is an embedding.*
2. *$\varphi\colon \mathcal{V} \to \mathcal{W}$ is a local embedding for any point $p \in \mathcal{V}$ and $\varphi\colon \mathcal{V} \to \varphi(\mathcal{V})$ is a homeomorphism.*

Proof. Standard.

□

1.5 Submersions

Theorem 1.21. *Let $\pi\colon \mathcal{V} \to \mathcal{W}$ be a differentiable map between smooth manifolds, let $p \in \mathcal{V}$ and let $q = \pi(p)$. The following conditions are equivalent:*

1. *The tangent linear map $\pi_*\colon T_p\mathcal{V} \to T_q\mathcal{W}$ is surjective.*
2. *There exist coordinate neighbourhoods V, W of p and q such that, representing each point by its coordinates, the map $\pi\colon V \to W$ is given by*

$$\pi(u_1, \ldots, u_n) \;=\; (u_1, \ldots, u_m) \quad , \quad m \leq n\ .$$

3. *There exists an open neighbourhood W of q in $\mathcal{W}$ and a differentiable section $\sigma\colon W \to \mathcal{V}$ (i.e., $\pi\sigma = Id$) such that $\sigma(q) = p$.*

Proof. Standard.

□

Definition. A differentiable map $\pi\colon \mathcal{V} \to \mathcal{W}$ between smooth manifolds is said to be a **submersion at a point** $p \in \mathcal{V}$ if it satisfies the above equivalent conditions, and it is said to be a **submersion** when so it is at any point of $\mathcal{V}$.

Examples. If $\mathcal{V}$ is a non-empty smooth manifold, then the second projection $\pi_2\colon \mathcal{V} \times \mathcal{W} \to \mathcal{W}$ is a surjective submersion.

Local diffeomorphisms are submersions.

The tangent bundle $T\mathcal{V} \to \mathcal{V}$ is a surjective submersion. In general, most fibre bundles (vector bundles, principal bundles) are surjective submersions.

Any family $\{U_i\}_{i\in I}$ of open subsets of a smooth manifold $\mathcal{V}$, when considered as the natural map $j\colon \coprod_i U_i \to \mathcal{V}$ (as we shall always do), is a submersion, and it is a surjective submersion whenever $\{U_i\}_{i\in I}$ is an open cover of $\mathcal{V}$.

Theorem 1.22 (Properties of surjective submersions).

1. *Diffeomorphisms are surjective submersions.*
2. *If $\pi\colon \mathcal{V} \to \mathcal{V}'$, $\pi'\colon \mathcal{V}' \to \mathcal{V}''$ are surjective submersions, then the composition $\pi' \circ \pi\colon \mathcal{V} \to \mathcal{V}''$ is a surjective submersion.*
3. *Let $\pi\colon \mathcal{V} \to \mathcal{W}$ be a surjective submersion. If $\varphi\colon T \to \mathcal{W}$ is a differentiable map, then*

$$\mathcal{V}_T \;:=\; \mathcal{V} \times_{\mathcal{W}} T \;:=\; \{(x,t) \in \mathcal{V} \times T\colon\; \pi(x) = \varphi(t)\}$$

   is a smooth submanifold of $\mathcal{V} \times T$, and the second projection $\pi_2\colon \mathcal{V}_T \to T$ is a surjective submersion.
4. *Any surjective submersion $\pi\colon \mathcal{V} \longrightarrow \mathcal{W}$ may be refined by some open cover $j\colon \coprod_i U_i \to \mathcal{W}$; that is to say, there exists a differentiable map $\sigma\colon \coprod U_i \to \mathcal{V}$ such that $\pi \circ \sigma = j$.*

5. *If* $\pi\colon \mathcal{V} \to \mathcal{W}$ *is a surjective submersion, then:*
   a) *a function* $f\colon \mathcal{W} \to \mathbb{R}$ *is differentiable* $\Leftrightarrow$ $f \circ \pi$ *is differentiable.*
   b) *a map* $\varphi\colon \mathcal{W} \to \mathcal{W}'$ *is differentiable* $\Leftrightarrow$ $\varphi \circ \pi$ *is differentiable.*
   c) *a map* $\varphi\colon \mathcal{W} \to \mathcal{W}'$ *is a submersion* $\Leftrightarrow$ $\varphi \circ \pi$ *is a submersion.*
   d) *a map* $\varphi\colon \mathcal{W} \to \mathcal{W}'$ *is a surjective submersion* $\Leftrightarrow$ $\varphi \circ \pi$ *is a surjective submersion.*
   e) *a subset* $Y \subseteq \mathcal{W}$ *is closed (respectively open)* $\Leftrightarrow$ $\pi^{-1}(Y)$ *is closed (respectively open) in* $\mathcal{V}$*.*
   f) *a subset* $Y \subseteq \mathcal{W}$ *is a smooth submanifold of* $\mathcal{W}$ $\Leftrightarrow$ $\pi^{-1}(Y)$ *is a smooth submanifold of* $\mathcal{V}$*.*

Proof. Properties 1 and 2 are obvious, while 4 follows readily from the existence of local sections of π.

(3) Let $(x,t) \in \mathcal{V} \times_{\mathcal{W}} T$ and let $y = \pi(x) = \varphi(t)$. Let us consider coordinate neighbourhoods $(U_x; x_1, \ldots, x_n)$, $(U_y; y_1, \ldots, y_m)$, $(U_t; t_1, \ldots, t_r)$ of x, y, t respectively. By 1.21, we may assume that the equations of π are $y_i = x_i$, $1 \le i \le m$. If the equations of φ are $y_i = \varphi_i(t_1, \ldots, t_r)$, $1 \le i \le m$, then

$$\mathcal{V}_T \cap (U_x \times U_t) = \{(x_1, \ldots, x_n, t_1, \ldots, t_r) \in U_x \times U_t\colon\ x_i = \varphi_i(t_1, \ldots, t_r)\}$$

is a smooth submanifold of $U_x \times U_t$ – by 1.17 – and $\pi_2\colon \mathcal{V}_T \cap (U_x \times U_t) \to U_t$ is a submersion at (x,t). Hence $\mathcal{V}_T$ is a smooth submanifold of $\mathcal{V} \times T$ and $\pi_2\colon \mathcal{V}_T \to T$ is a surjective submersion.

(5) The direct implications are clear (5.f follows from 3, since $\pi^{-1}(Y) = \mathcal{V}_Y$). To prove the converse implications, let us consider an open cover $j\colon \coprod U_i \to \mathcal{W}$ and a differentiable map $\sigma\colon \coprod U_i \to \mathcal{V}$ refining π, i.e., such that $j = \pi\sigma$ (this map σ exists by 4).

(5.a) If $f \circ \pi$ is differentiable, then so is $f \circ \pi \circ \sigma = \coprod f|_{U_i}$. Therefore f is differentiable.

A similar argument works for 5.b, 5.c, 5.d and 5.e.

(5.f) Let $y \in Y$ and let $\sigma\colon U \to \mathcal{V}$ be a local section of π defined on an open neighbourhood U of y in $\mathcal{W}$. Since

$$Y \cap U \ = \ \sigma^{-1}(\pi^{-1}Y) \ \simeq \ (\sigma U) \cap (\pi^{-1}Y) \ ,$$

it is enough to prove that $(\sigma U) \cap (\pi^{-1}Y)$ is a smooth submanifold of σU. Finally, $(\sigma U) \cap (\pi^{-1}Y)$ is a smooth submanifold because σU and $\pi^{-1}Y$ intersect transversally at any point $x \in (\sigma U) \cap (\pi^{-1}Y)$. In fact

$$T_x\mathcal{V} = T_x(\sigma U) + T_x(\pi^{-1}\pi x) \subseteq T_x(\sigma U) + T_x(\pi^{-1}Y) \ ,$$

hence $T_x\mathcal{V} = T_x(\sigma U) + T_x(\pi^{-1}Y)\,.$

□

Remarks. Properties 1, 2 and 3 may be restated asserting that surjective submersions define a Grothendieck topology on the category of smooth manifolds.

Since any open cover $\coprod U_i \to \mathcal{V}$ is a surjective submersion, this Grothendieck topology is stronger than the usual one, and property 4 shows that in fact both are equivalent, so explaining property 5, which states that the concepts of differentiable function, differentiable map, submersion, surjective submersion, open set, closed set and smooth submanifold are local concepts in this Grothendieck topology.

On the other hand, property 3 points that every surjective submersion $\pi\colon \mathcal{V} \to \mathcal{W}$ may be considered as a family $\{\mathcal{V}_y\}_{y\in\mathcal{W}}$ of smooth manifolds parametrized by $\mathcal{W}$, where $\mathcal{V}_y := \pi^{-1}(y)$. So $\mathcal{V}_T$ is just the family $\{\mathcal{V}_{\varphi(t)}\}_{t\in T}$, which is the reparametrization of the given family $\{\mathcal{V}_y\}_{y\in\mathcal{W}}$ by means of the differentiable map $\varphi\colon T \to \mathcal{W}$.

Theorems 1.19 and 1.21 are generalized by the following standard result:

Theorem 1.23. *Let $\varphi\colon \mathcal{V} \to \mathcal{W}$ be a differentiable map and let $p \in \mathcal{V}$. If the tangent linear map of φ has constant rank r at any point of some neighbourhood of p, then there exist local coordinate systems at p and $\varphi(p)$ such that*

$$\varphi(u_1, \dots, u_n) \;=\; (u_1, \dots, u_r, 0, \dots, 0)\ .$$

2 Differentiable Algebras

Even if we have introduced differentiable manifolds as certain pairs $(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$, where $\mathcal{O}_{\mathcal{V}}$ is the sheaf of differentiable functions on $\mathcal{V}$, it is a crucial fact that each manifold is determined by the ring $\mathcal{C}^\infty(\mathcal{V})$ of all global differentiable functions (when $\mathcal{V}$ is a Hausdorff space whose topology has a countable basis). In order to have a suitable perspective from which to consider these notes, it is important to see how $(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$ may be reconstructed from $\mathcal{C}^\infty(\mathcal{V})$. First, one recovers the set $\mathcal{V}$ as the set $\operatorname{Spec}_r \mathcal{C}^\infty(\mathcal{V})$ of all morphisms of $\mathbb{R}$-algebras $\delta\colon \mathcal{C}^\infty(\mathcal{V}) \to \mathbb{R}$, then its topology as the Gelfand topology, i.e., the initial topology for the maps $\hat{f}\colon \operatorname{Spec}_r \mathcal{C}^\infty(\mathcal{V}) \to \mathbb{R}$, $\hat{f}(\delta) := \delta(f)$, corresponding to elements $f \in \mathcal{C}^\infty(\mathcal{V})$. Finally, if U is any open set in $\mathcal{V}$, then $\mathcal{C}^\infty(U) = \mathcal{O}_{\mathcal{V}}(U)$ is just the localization (ring of fractions) $\mathcal{C}^\infty(\mathcal{V})_U$ of $\mathcal{C}^\infty(\mathcal{V})$ with respect to the multiplicative system of all global differentiable functions without zeros in U (a crucial fact proved in [37]):

$$\mathcal{C}^\infty(U) \;=\; \mathcal{C}^\infty(\mathcal{V})_U \;:=\; \{f/g\colon\; f, g \in \mathcal{C}^\infty(\mathcal{V}) :\; g(x) \neq 0, \forall x \in U\}\ .$$

This process is meaningful for an arbitrary $\mathbb{R}$-algebra A and thus we obtain a topological space $\operatorname{Spec}_r A$ endowed with a structural sheaf of rings $\widetilde{A}$, namely the sheaf associated to the presheaf $U \rightsquigarrow A_U$. When applied to $\mathcal{C}^\infty(\mathbb{R}^n)$, we obtain the local model $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ of all smooth manifolds and, when applied to other algebras, we obtain local building blocks for more general spaces.

Now let us look for the types of algebra that should be considered as *"algebras of differentiable functions on closed differentiable subspaces" of* $\mathbb{R}^n$. It is natural to assume that any differentiable function on the subspace is the restriction of some global differentiable function, so that the desired algebras must be certain quotients $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$, where $\mathfrak{a}$ is interpreted as the ideal of all differentiable functions vanishing on the subspace. Hence, the question is to fix the appropriate ideals. It seems natural to require differentiable subspaces to be defined by infinitesimal conditions (a condition is said to be infinitesimal whenever it involves only partial derivatives at a certain point). Now Whitney's spectral theorem states that ideals of $\mathcal{C}^\infty(\mathbb{R}^n)$ defined by infinitesimal conditions are just closed ideals with respect to the Fréchet topology of uniform convergence on compact sets of functions and their partial derivatives. Therefore we must consider *differentiable algebras:* quotients $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ by closed ideals. The main thesis of these notes is that differentiable algebras A provide local models $(\operatorname{Spec}_r A, \widetilde{A})$ for a reasonable concept of $\mathcal{C}^\infty$-differentiable space.

J.A. Navarro González and J.B. Sancho de Salas: LNM 1824, pp. 21–38, 2003.

2.1 Reconstruction of $\mathcal{V}$ from $\mathcal{C}^\infty(\mathcal{V})$

Let A be an $\mathbb{R}$-algebra (commutative with unity). An ideal $\mathfrak{m} \subset A$ is said to be **real** if the natural map $\mathbb{R} \to A/\mathfrak{m}$ is an isomorphism (in particular $\mathfrak{m}$ is a maximal ideal of A). The kernel of any morphism of $\mathbb{R}$-algebras $A \to \mathbb{R}$ is a real ideal, so we obtain a natural bijection between the set of all real ideals of A and the set of all morphisms of $\mathbb{R}$-algebras $A \to \mathbb{R}$. Let us briefly recall the spectral representation of algebras [14, 29, 37]:

Definition. The **real spectrum** of an $\mathbb{R}$-algebra A is the set

$$\mathrm{Spec}_r\, A := \mathrm{Hom}_{\mathbb{R}\text{-alg}}(A, \mathbb{R}) = \{\text{real ideals of } A\} .$$

If x is a point of $\mathrm{Spec}_r\, A$, then $\mathfrak{m}_x$ will denote the corresponding real ideal of A and $\delta_x \colon A \to \mathbb{R}$ will denote the corresponding morphism of $\mathbb{R}$-algebras. If $f \in A$, the **value** $f(x)$ of f at x is defined to be

$$f(x) := \delta_x(f) = \text{residue class of } f \text{ in } A/\mathfrak{m}_x = \mathbb{R} ,$$

so that any element $f \in A$ defines a real-valued function on $\mathrm{Spec}_r\, A$. Note that $\mathfrak{m}_x = \{f \in A\colon\ f(x) = 0\}$ and that $\delta_x \colon A \to \mathbb{R}$ is the evaluation map: $\delta_x(f) = f(x)$. When it is necessary, to avoid any confusion, we denote by $\hat{f}$ the function on $\mathrm{Spec}_r\, A$ induced by $f \in A$.

We consider the **Gelfand topology** on $\mathrm{Spec}_r\, A$, which is defined to be the smallest topology such that $\hat{f}\colon \mathrm{Spec}_r\, A \to \mathbb{R}$ is continuous for any $f \in A$. So we get a morphism of $\mathbb{R}$-algebras $A \to \mathcal{C}(\mathrm{Spec}_r\, A, \mathbb{R})$, $f \mapsto \hat{f}$. Generally, this spectral representation morphism is neither injective nor surjective.

Finally, any morphism of $\mathbb{R}$-algebras $\phi\colon A \to B$ defines a continuous map $\phi^*\colon \mathrm{Spec}_r\, B \to \mathrm{Spec}_r\, A$, $\phi^*(\delta) = \delta \circ \phi$.

Now let $\mathcal{V}$ be a smooth manifold. Each point $p \in \mathcal{V}$ defines a morphism of $\mathbb{R}$-algebras $\delta_p\colon \mathcal{C}^\infty(\mathcal{V}) \to \mathbb{R}$, $\delta_p(f) = f(p)$ or, equivalently, a real ideal

$$\mathfrak{m}_p = \{f \in \mathcal{C}^\infty(\mathcal{V})\colon\ f(p) = 0\}$$

of $\mathcal{C}^\infty(\mathcal{V})$. So we get a natural map $\delta\colon \mathcal{V} \to \mathrm{Spec}_r\, \mathcal{C}^\infty(\mathcal{V})$, $\delta(p) = \delta_p = \mathfrak{m}_p$.

Theorem 2.1. *If $\mathcal{V}$ is a separated smooth manifold whose topology has a countable basis, then the natural map $\delta\colon \mathcal{V} \to \mathrm{Spec}_r\, \mathcal{C}^\infty(\mathcal{V})$ is a homeomorphism.*

Proof. Let $p, q \in \mathcal{V}$. If $p \neq q$, then $\mathfrak{m}_p \neq \mathfrak{m}_q$ by 1.7. Hence δ is injective.

Let $\mathfrak{m}$ be a real ideal of $\mathcal{C}^\infty(\mathcal{V})$ and let us consider a sequence $\{K_n\}$ of compact sets in $\mathcal{V}$ such that $K_n \subseteq \overset{\circ}{K}_{n+1}$ and $\mathcal{V} = \bigcup_n K_n$. By 1.7 there exists a global differentiable function $0 \leq f_n \leq 1$ such that $f_n = 0$ on K_n and $f_n = 1$ outside $\overset{\circ}{K}_{n+1}$. Therefore $f = \sum_n f_n \in \mathcal{C}^\infty(\mathcal{V})$ and $f \geq n$ on $\mathcal{V} - K_{n+1}$, so that the level sets $f = r$ are compact for any real number r (since each one is contained in some K_n). Now, let $r \in \mathbb{R} = \mathcal{C}^\infty(\mathcal{V})/\mathfrak{m}$ be the residue class of f, so that $f - r \in \mathfrak{m}$.

If we have

$$\bigcap_{g\in\mathfrak{m}} \{g=0\} \;=\; \emptyset\,,$$

then $\emptyset = \{g_1 = 0\}\cap\ldots\cap\{g_m = 0\}\cap\{f = r\}$ for certain $g_1,\ldots,g_m \in \mathfrak{m}$ because $\{f = r\}$ is compact. It results that $h := g_1^2 + \ldots + g_m^2 + (f-r)^2 \in \mathfrak{m}$ does not vanish at any point of $\mathcal{V}$. Therefore $\mathfrak{m}$ contains an invertible function h, so contradicting the fact $\mathfrak{m} \neq \mathcal{C}^\infty(\mathcal{V})$. It follows the existence of some point $p\in\mathcal{V}$ such that $g(p) = 0$ for any $g \in \mathfrak{m}$, so that $\mathfrak{m} \subseteq \mathfrak{m}_p$ and, $\mathfrak{m}$ being a maximal ideal, we conclude that $\mathfrak{m} = \mathfrak{m}_p$. That is to say, δ is surjective.

Finally, identifying $\mathcal{V}$ with $\operatorname{Spec}_r \mathcal{C}^\infty(\mathcal{V})$ via δ, the topology of $\mathcal{V}$ coincides with the Gelfand topology. In fact, δ is clearly continuous and, given a closed set $Y \subseteq \mathcal{V}$, we consider the ideal $\mathfrak{p}_Y$ of all differentiable functions $f \in \mathcal{C}^\infty(\mathcal{V})$ vanishing on Y. By 1.7, we have

$$Y \;=\; \{x\in\mathcal{V}\colon\; f(x) = 0\;,\;\forall f \in \mathfrak{p}_Y\}$$

and we conclude that Y is a closed set for the Gelfand topology.

□

Remark. For any subset I of an $\mathbb{R}$-algebra A, the **zero-set** of I is defined to be

$$(I)_0 \;:=\; \{x \in \operatorname{Spec}_r A\colon\; f(x) = 0,\;\forall f\in I\}$$

and it is easy to check that these subsets are the closed sets of a topology on $\operatorname{Spec}_r A$ called the **Zariski topology**. The Gelfand topology is always finer than this Zariski topology, but the proof of the former theorem shows that both coincide on $\operatorname{Spec}_r \mathcal{C}^\infty(\mathcal{V}) = \mathcal{V}$.

Lemma 2.2. *Let $\varphi\colon \mathcal{V}\to\mathcal{W}$ be a continuous map between separated smooth manifolds. If $f\circ\varphi \in \mathcal{C}^\infty(\mathcal{V})$ for any $f\in\mathcal{C}^\infty(\mathcal{W})$, then φ is a differentiable map.*

Proof. Let U be an open set in $\mathcal{W}$ and let $V = \varphi^{-1}U$. If $f\in\mathcal{C}^\infty(U)$, then we have to prove that $f\circ\varphi\in\mathcal{C}^\infty(V)$. Let $p\in V$. By 1.6, f coincides with some global differentiable function $F\in\mathcal{C}^\infty(\mathcal{W})$ on a neighbourhood of $\varphi(p)$. By hypothesis $F\circ\varphi\in\mathcal{C}^\infty(\mathcal{V})$, so that $f\circ\varphi$ coincides with the differentiable function $F\circ\varphi$ on a neighbourhood of p and we conclude that $f\circ\varphi$ is a differentiable function.

□

Let $\operatorname{Hom}(\mathcal{V},\mathcal{W})$ be the set of all differentiable maps $\varphi\colon\mathcal{V}\to\mathcal{W}$. Recall that $\varphi\in\operatorname{Hom}(\mathcal{V},\mathcal{W})$ defines a morphism $\varphi^*\colon\mathcal{C}^\infty(\mathcal{W})\to\mathcal{C}^\infty(\mathcal{V})$, $\varphi^* f = f\circ\varphi$.

Theorem 2.3. *Let $\mathcal{V}$, $\mathcal{W}$ be separated smooth manifolds whose topologies have countable bases. We have a natural bijection*

$$\begin{aligned}\operatorname{Hom}(\mathcal{V},\mathcal{W}) \;&=\; \operatorname{Hom}_{\mathbb{R}\text{-alg}}(\mathcal{C}^\infty(\mathcal{W}),\mathcal{C}^\infty(\mathcal{V}))\\ \varphi \;&\mapsto\; \varphi^*\,.\end{aligned}$$

Proof. Let us define the inverse correspondence: Any morphism of $\mathbb{R}$-algebras $\phi\colon \mathcal{C}^\infty(\mathcal{W}) \to \mathcal{C}^\infty(\mathcal{V})$ defines a continuous map

$$\begin{gathered}\phi^*\colon \mathcal{V} = \operatorname{Spec}_r \mathcal{C}^\infty(\mathcal{V}) \longrightarrow \operatorname{Spec}_r \mathcal{C}^\infty(\mathcal{W}) = \mathcal{W} \\ \phi^*(\delta_p) \;=\; \delta_p \circ \phi\end{gathered}$$

i.e., $\phi^*(p) = q$ when $\delta_q = \delta_p \circ \phi$. We must prove that $\phi^{**} = \phi$ (so that ϕ^* is differentiable by 2.2) and $\varphi^{**} = \varphi$. Now, by definition we have

$$\begin{gathered}\varphi^{**}(p) = \varphi^{**}(\delta_p) = \delta_p \circ \varphi^* = \delta_{\varphi(p)} = \varphi(p)\,, \\ (\phi^{**}f)(p) = f(\phi^* p) = \delta_{\phi^* p}(f) = (\delta_p \circ \phi)(f) = \phi(f)(p)\,.\end{gathered}$$

□

Proposition 2.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be separated smooth manifolds whose topologies have countable bases. A differentiable map $\varphi\colon \mathcal{W} \to \mathcal{V}$ is a closed embedding if and only if the morphism $\varphi^*\colon \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{W})$ is surjective.*

Proof. If $\varphi\colon \mathcal{W} \to \mathcal{V}$ is a closed embedding, we may assume that $\mathcal{W}$ is a closed smooth submanifold of $\mathcal{V}$ and that φ is the inclusion map. Let $f \in \mathcal{C}^\infty(\mathcal{W})$. By definition of the induced sheaf, any point $p \in \mathcal{W}$ has an open neighbourhood V_p in $\mathcal{V}$ such that $f|_{\mathcal{W}\cap V_p} = F_p|_{\mathcal{W}\cap V_p}$ for some $F_p \in \mathcal{C}^\infty(V_p)$. Let $\{h_p, h\}_{p\in\mathcal{W}}$ be a partition of unity subordinated to the open cover $\{V_p, \mathcal{V} - \mathcal{W}\}_{p\in\mathcal{W}}$ and let $F = \sum_p h_p F_p \in \mathcal{C}^\infty(\mathcal{V})$. It is clear that $f = F|_{\mathcal{W}} = \varphi^* F$.

Conversely, if $\varphi^*\colon \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{W})$ is surjective then

$$\varphi^*\colon \mathcal{O}_{\mathcal{V},\varphi p} \;=\; \mathcal{C}^\infty(\mathcal{V})/\mathfrak{n}_{\varphi p} \;\longrightarrow\; \mathcal{C}^\infty(\mathcal{W})/\mathfrak{n}_p \;=\; \mathcal{O}_{\mathcal{W},p} \qquad \text{(see 1.6)}$$

also is surjective for any $p \in \mathcal{W}$. Hence

$$\varphi_*\colon T_p\mathcal{W} \;=\; \operatorname{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{W},p}, \mathbb{R}) \;\longrightarrow\; \operatorname{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{V},\varphi p}, \mathbb{R}) \;=\; T_{\varphi p}\mathcal{V}$$

is injective, i.e., φ is a local embedding at any $p \in \mathcal{W}$. Now, let I be the kernel of $\varphi^*\colon \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{W})$. It is direct to show that

$$\varphi\colon \mathcal{W} \;=\; \operatorname{Hom}_{\mathbb{R}\text{-alg}}(\mathcal{C}^\infty(\mathcal{W}), \mathbb{R}) \;\longrightarrow\; \operatorname{Hom}_{\mathbb{R}\text{-alg}}(\mathcal{C}^\infty(\mathcal{V}), \mathbb{R}) \;=\; \mathcal{V}$$

induces a homeomorphism of $\mathcal{W}$ onto the zero-set

$$(I)_0 \;=\; \{x \in \mathcal{V} : f(x) = 0, \forall f \in I\}\,.$$

Now 1.20 let us conclude that φ is an embedding.

□

2.2 Regular Ideals

Let $\mathcal{V}$ be a separated smooth manifold whose topology has a countable basis. Given a closed subset Y, we denote by $\mathfrak{p}_Y$ the ideal of $\mathcal{C}^\infty(\mathcal{V})$ of all differentiable functions vanishing on Y.

If Y is a closed smooth submanifold we have $\mathcal{C}^\infty(Y) = \mathcal{C}^\infty(\mathcal{V})/\mathfrak{p}_Y$ by 2.4. Our purpose in this section is to characterize when a closed subset Y is a smooth submanifold in terms of the ideal $\mathfrak{p}_Y$.

Given a point $x \in \mathcal{V}$ we denote by $\mathfrak{p}_{Y,x}$ the image of $\mathfrak{p}_Y$ by the epimorphism $\mathcal{C}^\infty(\mathcal{V}) \to \mathcal{O}_x$. In other words, $\mathfrak{p}_{Y,x}$ is the ideal of $\mathcal{O}_x$ of all germs of differentiable functions f, defined on some open neighbourhood U_f of x, such that $f = 0$ on $Y \cap U_f$. Note that $\mathfrak{p}_{Y,x} = \mathcal{O}_x$ when $x \notin Y$.

More generally, if I is an ideal of $\mathcal{C}^\infty(\mathcal{V})$, then we denote by I_x the image of I by the epimorphism $\mathcal{C}^\infty(\mathcal{V}) \to \mathcal{O}_x$.

Lemma 2.5. *Let $\mathfrak{p}_Y$ be the ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$ of all differentiable functions vanishing on $Y = \{p \in \mathbb{R}^n : x_1(p) = \cdots = x_r(p) = 0\}$. Then $\mathfrak{p}_Y = (x_1, \ldots, x_r)$.*

Proof. Let $f \in \mathfrak{p}_Y$. For any point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ we may consider the function

$$g(t) \;=\; f(tx_1, \ldots, tx_r, x_{r+1}, \ldots, x_n) \quad , \quad t \in [0,1] \,.$$

Since $f(0, \ldots, 0, x_{r+1}, \ldots, x_n) = 0$, we have

$$\begin{aligned} f(x) = g(1) - g(0) &= \int_0^1 g'(t)\,\mathrm{d}t \\ &= \sum_{i=1}^r \int_0^1 x_i \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_r, x_{r+1}, \ldots, x_n)\,\mathrm{d}t \\ &= \sum_{i=1}^r x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_r, x_{r+1}, \ldots, x_n)\,\mathrm{d}t = \sum_{i=1}^r x_i h_i(x) \end{aligned}$$

hence $f \in (x_1, \ldots, x_r)$.

Finally, the inclusion $(x_1, \ldots, x_r) \subseteq \mathfrak{p}_Y$ is obvious.

□

Remark 2.6. We have an equality $\mathfrak{p}_{Y,p} = (x_1, \ldots, x_r)_p$ of ideals in the ring of germs $\mathcal{O}_p$ for any point $p \in \mathbb{R}^n$.

Let $\mathcal{V}$ be a separated smooth manifold whose topology has a countable basis. Given differentiable functions $f_1, \ldots, f_r \in \mathcal{C}^\infty(\mathcal{V})$ let $Y = \{x \in \mathcal{V} : f_1(x) = \cdots = f_r(x) = 0\}$. Let us assume that $\mathrm{d}_x f_1, \ldots, \mathrm{d}_x f_r$ are linearly independent at any $x \in Y$, so that Y is a closed smooth submanifold of $\mathcal{V}$ by 1.17. Recall that $\mathfrak{p}_Y$ denotes the ideal of $\mathcal{C}^\infty(\mathcal{V})$ of all differentiable functions vanishing on Y.

Proposition 2.7. *With the previous notations, we have* $\mathfrak{p}_Y = (f_1, \ldots, f_r)$.

Proof. Let us show that $\mathfrak{p}_{Y,p} = (f_1, \ldots, f_r)_p$ for any $p \in \mathcal{V}$ (equality of ideals in $\mathcal{O}_{\mathcal{V},p}$). If $p \notin Y$, the equality is clear since both ideals coincide with $\mathcal{O}_{\mathcal{V},p}$. If $p \in Y$, then $\mathrm{d}_p f_1, \ldots, \mathrm{d}_p f_r$ are linearly independent, hence $f_1, \ldots, f_r$ are part of a coordinate system at p. By 2.6 we obtain that $\mathfrak{p}_{Y,p} = (f_1, \ldots, f_r)_p$.

Now we prove that $\mathfrak{p}_Y = (f_1, \ldots, f_r)$. If $f \in \mathfrak{p}_Y$ then $f_p \in (f_1, \ldots, f_r)_p$ for any $p \in \mathcal{V}$ and, using a suitable partition of unity, it is easy to conclude that $f \in (f_1, \ldots, f_r)$. The inclusion $(f_1, \ldots, f_r) \subseteq \mathfrak{p}_Y$ is obvious.

□

Definition. Let $\mathcal{V}$ be a separated smooth manifold whose topology has a countable basis. An ideal I of $\mathcal{C}^\infty(\mathcal{V})$ is said to be **regular** if, for any $p \in (I)_0$, the ideal I_p is generated by some germs $f_1, \ldots, f_r$ such that $\mathrm{d}_p f_1, \ldots, \mathrm{d}_p f_r$ are linearly independent.

Lemma 2.8. *Let* Y, Z *be closed subsets of a separated smooth manifold* $\mathcal{V}$ *whose topology has a countable basis. If* $\mathfrak{p}_{Y,x} = \mathfrak{p}_{Z,x}$ *then there exists a neighbourhood* U *of* x *such that* $Y \cap U = Z \cap U$.

Proof. We shall show in 2.10 the existence of a global differentiable function f on $\mathcal{V}$ such that $(f)_0 = Y$. Since $f_x \in \mathfrak{p}_{Y,x} = \mathfrak{p}_{Z,x}$ it results that f vanishes on $Z \cap U$ for a certain neighbourhood U of x, hence $Z \cap U \subseteq Y \cap U$. Analogously we prove the existence of a neighbourhood V of x such that $Y \cap V \subseteq Z \cap V$, so that $U \cap V$ is the desired neighbourhood.

□

Proposition 2.9. *Let* Y *be a closed subset of a separated smooth manifold* $\mathcal{V}$ *whose topology has a countable basis.* Y *is a smooth submanifold of* $\mathcal{V}$ *if and only if* $\mathfrak{p}_Y$ *is a regular ideal of* $\mathcal{C}^\infty(\mathcal{V})$.

Proof. Let Y be a closed submanifold of $\mathcal{V}$. For any $p \in Y = (\mathfrak{p}_Y)_0$ there exists a coordinate neighbourhood $(U; u_1, \ldots, u_n)$ such that

$$Y \cap U \ = \ \{x \in U : u_1(x) = \cdots = u_r(x) = 0\} \ ,$$

hence $\mathfrak{p}_{Y,p} = (u_1, \ldots, u_r)_p$ by 2.7, and $\mathfrak{p}_Y$ is a regular ideal.

Conversely, let us assume that $\mathfrak{p}_Y$ is a regular ideal. If $p \in Y = (\mathfrak{p}_Y)_0$, we have $\mathfrak{p}_{Y,p} = (f_1, \ldots, f_r)_p$, where $\mathrm{d}_p f_1, \ldots, \mathrm{d}_p f_r$ are linearly independent. This last condition implies the existence of an open neighbourhood U of p such that the subset $Z = \{x \in U : f_1(x) = \cdots = f_r(x) = 0\}$ is a closed smooth submanifold of U. By 2.7 we have $\mathfrak{p}_{Z,p} = (f_1, \ldots, f_r)_p = \mathfrak{p}_{Y,p}$, hence Y and Z coincide in a neighbourhood of p (by 2.8) and we conclude that Y is a smooth submanifold of $\mathcal{V}$.

□

2.3 Fréchet Topology of $\mathcal{C}^\infty(\mathcal{V})$

Let us recall the notion of Fréchet vector space. A locally convex vector space is said to be a **Fréchet vector space** if it is metrizable (its topology is separated and it may be defined by a countable family of seminorms) and complete. We shall use the following basic facts:

1. *Let F be a closed vector subspace of a Fréchet vector space E. Then F and E/F (endowed with the quotient topology) are Fréchet vector spaces.*
2. *Any surjective continuous linear map between Fréchet vector spaces is an open map.*

Definition. A **locally m-convex algebra** is defined to be an $\mathbb{R}$-algebra (commutative with unity) A endowed with a topology defined by a family $\{q_i\}$ of submultiplicative seminorms: $q_i(ab) \le q_i(a)q_i(b)$.

If A is a locally m-convex algebra, then the addition $A \times A \xrightarrow{+} A$ and the product $A \times A \xrightarrow{\cdot} A$ are continuous operations. Moreover, the inversion map $A^* \to A^*$, $a \mapsto a^{-1}$, is continuous on the set A^* of all invertible elements; hence A^* is a topological group.

If I is an ideal of a locally convex m-algebra A, then A/I is a locally m-convex algebra with the quotient topology: If $\{q_i\}$ is a fundamental system of submultiplicative seminorms of A, then the topology of A/I is defined by the submultiplicative seminorms $\overline{q}_i([a]) = \inf_{b\in I} q_i(a+b)$.

The canonical projection $\pi\colon A \to A/I$ is an open map.

The closure $\overline{I}$ of an ideal I is again an ideal of A.

We say that a locally m-convex algebra is **complete** when so it is as a locally convex space (hence it is separated by definition).

Definition. A locally m-convex algebra is said to be a **Fréchet algebra** if it is metrizable and complete.

If I is a closed ideal of a Fréchet algebra A, then A/I is a Fréchet algebra.

Let $\mathcal{V}$ be a separated smooth manifold whose topology has a countable basis, and let us consider a countable family $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets such that $\mathcal{V} = \bigcup_i \overset{\circ}{K}_i$ and each compact set K_i is contained in some coordinate open set $(U_i; u_1, \ldots, u_n)$. The **usual topology** of $\mathcal{C}^\infty(\mathcal{V})$ is defined by the following submultiplicative seminorms (see [27] IV.4.2)

$$p_{i,j}(f) \;:=\; \max\, 2^j \left| \frac{\partial^{|\alpha|} f}{\partial u^\alpha}(p) \right| ,$$

where the maximum is considered over all points $p \in K_i$ and all orders of derivation $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $|\alpha| = \alpha_1 + \ldots + \alpha_n \le j$. Sometimes it will be convenient to replace this family of seminorms $\{p_{i,j}\}_{i,j\in\mathbb{N}}$ by an equivalent increasing family $\{q_i\}_{i\in\mathbb{N}}$, where $q_i(f) := \max\{p_{1,i}(f), \ldots, p_{i,i}(f)\}$. This topology is independent of the different choices.

It is a basic fact that $\mathcal{C}^\infty(\mathcal{V})$ is complete, so that it is a Fréchet algebra. Moreover, this topology provides the unique structure of Fréchet algebra that may be defined on the algebra $\mathcal{C}^\infty(\mathcal{V})$, according to a theorem of Michael [29].

Finally, note that for any differentiable map $\varphi\colon \mathcal{W} \to \mathcal{V}$ the corresponding morphism $\varphi^*\colon \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{W})$, $\varphi^*(f) = f \circ \varphi$, is continuous.

Proposition 2.10. *Let $\mathcal{V}$ be a separated smooth manifold whose topology has a countable basis. Any closed subset $Y \subseteq \mathcal{V}$ is the zero-set of some global differentiable function.*

Proof. Let $\{K_i\}$ be a countable family of compact sets such that $\mathcal{V} - Y = \bigcup_i K_i$. Let $f_i \in \mathcal{C}^\infty(\mathcal{V})$ be such that $0 \le f_i \le 1$, $f_i(K_i) = 1$ and $f_i(Y) = 0$. Then Y is the zero-set of the global differentiable function

$$f \;=\; \sum_{i=1}^{\infty} 2^{-i} \frac{f_i}{1 + q_i(f_i)} \;.$$

□

Given an open set U in $\mathcal{V}$, we shall denote by $\mathcal{C}^\infty(\mathcal{V})_U$ the localization (ring of fractions, [1] Chapter 3) of $\mathcal{C}^\infty(\mathcal{V})$ with respect to the multiplicative set of all functions in $\mathcal{C}^\infty(\mathcal{V})$ without zeros in U. So, elements in $\mathcal{C}^\infty(\mathcal{V})_U$ are equivalence classes of fractions $[g/h]$, where $g, h \in \mathcal{C}^\infty(\mathcal{V})$ and h has not zeros in U.

Localization Theorem for differentiable functions ([37]). *Let $\mathcal{V}$ be a separated smooth manifold whose topology has a countable basis and let U be an open subset. For any differentiable function f on U, there exist global functions $g, h \in \mathcal{C}^\infty(\mathcal{V})$ such that h does not vanish at any point of U and*

$$f \;=\; \frac{g}{h} \qquad \text{on } U.$$

That is to say,

$$\mathcal{C}^\infty(U) \;=\; \mathcal{C}^\infty(\mathcal{V})_U \;.$$

Proof. Let $\{K_i\}$ be a countable family of compact sets such that

$$K_i \subseteq \overset{\circ}{K}_{i+1} \qquad , \qquad U = \textstyle\bigcup_i K_i \;.$$

Let $g_i \in \mathcal{C}^\infty(\mathcal{V})$ such that $0 \le g_i \le 1$, $g_i(K_i) = 1$ and $\operatorname{Supp} g_i \subseteq U$, so that $g_i f \in \mathcal{C}^\infty(\mathcal{V})$ when extended by 0. It is easy to check that

$$g = \sum_{i=1}^{\infty} 2^{-i} \frac{g_i f}{1 + q_i(g_i) + q_i(g_i f)} \;,$$

$$h = \sum_{i=1}^{\infty} 2^{-i} \frac{g_i}{1 + q_i(g_i) + q_i(g_i f)} \;,$$

satisfy the required conditions.

Therefore, the natural map $\mathcal{C}^\infty(\mathcal{V})_U \to \mathcal{C}^\infty(U)$, $[g/h] \mapsto g/h$, is surjective. Let us show that it is also injective: If $g/h = 0$ in $\mathcal{C}^\infty(U)$, then $g = 0$ on U. By 2.10 there exists $d \in \mathcal{C}^\infty(\mathcal{V})$ such that $(d)_0 = \mathcal{V} - U$; hence $gd = 0$, so that $[g/h] = [gd/hd] = 0$ in $\mathcal{C}^\infty(\mathcal{V})_U$.

□

Now, we deal with closed ideals of $\mathcal{C}^\infty(\mathcal{V})$ with respect to the usual Fréchet topology.

Proposition 2.11. *Let $\mathfrak{m}_{\mathcal{V},p}$ be the ideal of $\mathcal{C}^\infty(\mathcal{V})$ of all differentiable functions vanishing at a point $p \in \mathcal{V}$. The ideal $\mathfrak{m}_{\mathcal{V},p}^{r+1}$ is closed for any $r \geq 0$.*

Proof. By 1.11 we have $\mathfrak{m}_{\mathcal{V},p}^{r+1} = \{f \in \mathcal{C}^\infty(\mathcal{V}) : j_p^r f = 0\}$ and the result follows easily.

□

Definition. Let X be a closed set in $\mathcal{V}$. The **Whitney ideal** of X in $\mathcal{V}$ is defined to be the ideal W_X of all differentiable functions $f \in \mathcal{C}^\infty(\mathcal{V})$ with Taylor expansion $j_x f = 0$ at any point $x \in X$. It is a closed ideal because

$$W_X = \bigcap_{x\in X} W_x = \bigcap_{x,r} \mathfrak{m}_{\mathcal{V},x}^{r+1} \qquad (x \in X,\ r \in \mathbb{N}) .$$

Let $\mathfrak{m}_p$ be the maximal ideal in the ring of germs $\mathcal{O}_p$. Let us consider the $\mathfrak{m}_p$-adic completion of $\mathcal{O}_p$ (see 1.10)

$$\widehat{\mathcal{O}}_p := \varprojlim_r (\mathcal{O}_p/\mathfrak{m}_p^{r+1}) = \mathbb{R}[[u_1 - p_1, \ldots, u_n - p_n]]$$

endowed with the projective limit topology. It is easy to check that $\widehat{\mathcal{O}}_p$ is a Fréchet algebra and that the "Taylor expansion" map

$$\mathcal{C}^\infty(\mathcal{V}) \longrightarrow \widehat{\mathcal{O}}_p \quad , \quad f \mapsto j_p f$$

is continuous. By a result of Borel ([26] Chapter I, §4), the above map is surjective. Its kernel is obviously the Whitney ideal W_p of all flat functions at p, hence $\mathcal{C}^\infty(\mathcal{V})/W_p = \widehat{\mathcal{O}}_p$. This continuous isomorphism is a homeomorphism since both topologies are Fréchet. In conclusion, we may state Borel's result as follows:

Borel's theorem. *Let $\mathcal{V}$ be a separated smooth manifold whose topology has a countable basis. For any point $p \in \mathcal{V}$ we have an algebraic and topological isomorphism*

$$\mathcal{C}^\infty(\mathcal{V})/W_p = \widehat{\mathcal{O}}_p .$$

The following central result provides a characterization of closed ideals.

Whitney's spectral theorem. *Let $\mathcal{V}$ be a separated smooth manifold whose topology has a countable basis. Let $\mathfrak{a}$ be an ideal of $\mathcal{C}^\infty(\mathcal{V})$. The closure of $\mathfrak{a}$ in $\mathcal{C}^\infty(\mathcal{V})$ is the ideal of all differentiable functions whose Taylor expansion at any point of $\mathcal{V}$ coincides with the Taylor expansion of some function in $\mathfrak{a}$:*

$$\bar{\mathfrak{a}} = \{f \in \mathcal{C}^\infty(\mathcal{V}) :\ j_x f \in j_x(\mathfrak{a}) \text{ for any } x \in \mathcal{V}\} .$$

Proof. [26] Chapter II.

□

Remark. Any formal power series with non-zero leading coefficient is invertible in $\widehat{\mathcal{O}}_x$. This implies that $j_x(\mathfrak{a}) = \widehat{\mathcal{O}}_x$ for any $x \notin (\mathfrak{a})_0$. So we may put

$$\bar{\mathfrak{a}} \;=\; \{f \in \mathcal{C}^\infty(\mathcal{V}) :\; j_x f \in j_x(\mathfrak{a}) \text{ for any } x \in (\mathfrak{a})_0\} \;.$$

Moreover, $j_x(\mathfrak{a})$ is an ideal of the noetherian local ring $\widehat{\mathcal{O}}_x$; hence, by Krull's theorem (see [1] 10.20) the $\mathfrak{m}_x$-adic topology of $\widehat{\mathcal{O}}_x / j_x(\mathfrak{a})$ is separated, so that $j_x(\mathfrak{a}) = \bigcap_r (j_x(\mathfrak{a}) + \mathfrak{m}_x^r)$. Therefore, the former condition $j_x f \in j_x(\mathfrak{a})$ may be replaced by the following weaker condition: $j_x^r f \in j_x^r(\mathfrak{a})$ for any $r \in \mathbb{N}$.

2.4 Differentiable Algebras

Definition. We say that an $\mathbb{R}$-algebra A is a **differentiable algebra** if it is (algebraically) isomorphic to a quotient

$$A \;\simeq\; \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$$

for some natural number n and some closed ideal $\mathfrak{a}$ of $\mathcal{C}^\infty(\mathbb{R}^n)$. Such isomorphism is called a **presentation** of A.

A map $A \to B$ between differentiable algebras is said to be a **morphism of differentiable algebras** when it is a morphism of $\mathbb{R}$-algebras.

Let us determinate the real spectrum of any differentiable algebra.

Lemma 2.12. *Let I be an ideal of an $\mathbb{R}$-algebra A. There exists a natural homeomorphism*

$$\operatorname{Spec}_r A/I = (I)_0 := \{p \in \operatorname{Spec}_r A : f(p) = 0, \forall f \in I\} \;.$$

Proof. Let us consider the quotient map $\pi\colon A \to A/I$. It is clear that a morphism $\delta_p\colon A \to \mathbb{R}$ factors through A/I if and only if $\delta_p(I) = 0$ or, equivalently, $p \in (I)_0$. Therefore, the continuous map

$$\begin{array}{ccc} \operatorname{Spec}_r A/I = \operatorname{Hom}_{\mathbb{R}\text{-alg}}(A/I, \mathbb{R}) & \longrightarrow & \operatorname{Hom}_{\mathbb{R}\text{-alg}}(A, \mathbb{R}) = \operatorname{Spec}_r A \\ \delta & \mapsto & \delta \circ \pi \end{array}$$

defines a bijection $\operatorname{Spec}_r A/I = (I)_0$. It is easy to check that this bijection is a homeomorphism.

□

Proposition 2.13. *For any quotient algebra $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ we have a homeomorphism*

$$\operatorname{Spec}_r A \;=\; (\mathfrak{a})_0 \;:=\; \{p \in \mathbb{R}^n : f(p) = 0, \forall f \in \mathfrak{a}\} \;.$$

Proof. By 2.1 we have $\operatorname{Spec}_r \mathcal{C}^\infty(\mathbb{R}^n) = \mathbb{R}^n$. The result follows from the previous lemma.

□

Remark 2.14. Any closed set in $\operatorname{Spec}_r A$ is a zero-set, since so is any closed set in $\mathbb{R}^n$; hence the Gelfand and Zariski topologies in $\operatorname{Spec}_r A$ coincide for any differentiable algebra A.

Proposition 2.15. *Let A be a differentiable algebra. An element $a \in A$ is invertible if and only if it does not vanish at any point of* $\operatorname{Spec}_r A$.

Proof. Let $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$. If $a = [f]$ does not vanish at any point of $\operatorname{Spec}_r A$, then $(f)_0 \cap (\mathfrak{a})_0 = \emptyset$. By 1.7, there exists a differentiable function $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $g = 0$ on a neighbourhood of $(\mathfrak{a})_0$ and $g = 1$ on $(f)_0$, so that $f^2 + g^2$ does not vanish at any point of $\mathbb{R}^n$ and it is invertible in $\mathcal{C}^\infty(\mathbb{R}^n)$. By Whitney's spectral theorem $g \in \mathfrak{a}$, so that $[f^2]$, hence $[f]$, is invertible in A.

□

Definition. Let us introduce the notion of Taylor expansion for elements in a differentiable algebra. Let $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ be a differentiable algebra. Given a point $p \in \operatorname{Spec}_r A = (\mathfrak{a})_0$, we denote by $\mathfrak{m}_p$ the corresponding real ideal of A. Recall that we also denote by $\mathfrak{m}_p$ the respective ideals of $\mathcal{C}^\infty(\mathbb{R}^n)$ and $\mathcal{O}_p$ of all differentiable functions vanishing at p. The **Taylor expansion of order** r (or r-jet) of an element $a \in A$ at p is defined to be the equivalence class

$$j^r_p a \ := \ [a]_r \in A/\mathfrak{m}_p^{r+1} \,.$$

Let us consider the $\mathfrak{m}_p$-adic completion of A,

$$\widehat{A}_p \ := \ \varprojlim_r \, (A/\mathfrak{m}_p^{r+1})$$

and the natural morphism

$$j_p \colon A \ \longrightarrow \ \widehat{A}_p \quad , \quad a \mapsto j_p a := \varprojlim \, [a]_r .$$

The element $j_p a \in \widehat{A}_p$ is called the **Taylor expansion** (or ∞-jet) of a at p.

Let us compute $\widehat{A}_p$. Let $\widehat{\mathfrak{a}}_p := j_p(\mathfrak{a})$ be the image of $\mathfrak{a}$ by the map

$$j_p \colon \mathcal{C}^\infty(\mathbb{R}^n) \ \longrightarrow \ \widehat{\mathcal{O}}_p \ = \ \mathbb{R}[[x_1 - p_1, \dots, x_n - p_n]],$$

i.e., $\widehat{\mathfrak{a}}_p$ is the ideal of $\widehat{\mathcal{O}}_p$ of all Taylor expansions of functions in $\mathfrak{a}$.

Proposition 2.16. *With the previous notations, we have an isomorphism*

$$\widehat{A}_p \ = \ \widehat{\mathcal{O}}_p/\widehat{\mathfrak{a}}_p \,.$$

Proof. Elements of $\widehat{A}_p = \varprojlim A/\mathfrak{m}_p^{r+1}$ are $(a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots)$ where $a_r \in \mathfrak{m}_p^r$ (an analogous statement holds for $\widehat{\mathcal{O}}_p$), so that the natural morphism

$$\widehat{\mathcal{O}}_p \;=\; \varprojlim_r \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{m}_{\mathbb{R}^n,p}^{r+1} \;\longrightarrow\; \varprojlim_r A/\mathfrak{m}_p^{r+1} \;=\; \widehat{A}_p$$

is surjective. The kernel of the natural morphism $\widehat{\mathcal{O}}_p \to A/\mathfrak{m}_p^{r+1}$ is just the ideal $\widehat{\mathfrak{a}}_p + \widehat{\mathfrak{m}}_p^{r+1}$, and it follows that the kernel of $\widehat{\mathcal{O}}_p \to \widehat{A}_p$ is $\bigcap_r(\widehat{\mathfrak{a}}_p + \widehat{\mathfrak{m}}_p^{r+1})$. Now, since the formal power series ring $\widehat{\mathcal{O}}_p$ is a local noetherian ring, by a standard result of commutative algebra (Krull's theorem, see [1] 10.20), we have $\widehat{\mathfrak{a}}_p = \bigcap_r(\widehat{\mathfrak{a}}_p + \widehat{\mathfrak{m}}_p^{r+1})$, and we conclude that $\widehat{A}_p = \widehat{\mathcal{O}}_p/\widehat{\mathfrak{a}}_p$.

□

Proposition 2.17. *Let A be a differentiable algebra. An element $a \in A$ is zero if and only if its Taylor expansion $j_p a$ at any point $p \in \operatorname{Spec}_r A$ vanishes.*

Proof. Let $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ be a presentation and let $a = [f] \in A$. By Whitney's spectral theorem we have $a = 0$ (i.e., $f \in \mathfrak{a}$) if and only if $j_p f \in j_p(\mathfrak{a}) = \widehat{\mathfrak{a}}_p$ for any $p \in (\mathfrak{a})_0 = \operatorname{Spec}_r A$. This last condition is equivalent, by the previous proposition, to the condition $j_p a = 0$ for any $p \in \operatorname{Spec}_r A$.

□

Corollary 2.18. *For any differentiable algebra A we have*

$$\operatorname{Spec}_r A = \emptyset \;\Leftrightarrow\; A = 0.$$

Let $\phi^*\colon \operatorname{Spec}_r B \longrightarrow \operatorname{Spec}_r A$ be the map induced by a morphism of differentiable algebras $\phi\colon A \to B$. Given a point $q \in \operatorname{Spec}_r B$, let $p = \phi^*(q)$, i.e., $\delta_p = \delta_q \circ \phi$. Note that $\phi(\mathfrak{m}_p) \subseteq \mathfrak{m}_q$, so that we have a morphism

$$\phi_r\colon A/\mathfrak{m}_p^{r+1} \longrightarrow B/\mathfrak{m}_q^{r+1} \qquad , \qquad [a] \mapsto [\phi(a)]\ .$$

Taking projective limits we obtain a morphism $\widehat{\phi}\colon \widehat{A}_p \to \widehat{B}_q$ and an obvious commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \phi\ } & B \\ \big\downarrow{\scriptstyle j_p} & & \big\downarrow{\scriptstyle j_q} \\ \widehat{A}_p & \xrightarrow{\ \widehat{\phi}\ } & \widehat{B}_q \end{array}$$

The following basic result shows that differentiable algebras have a similar behaviour than finitely generated $\mathbb{R}$-algebras.

Proposition 2.19. *Let $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ and $B = \mathcal{C}^\infty(\mathbb{R}^m)/\mathfrak{b}$ be differentiable algebras and let $\phi, \phi'\colon A \to B$ be morphisms of $\mathbb{R}$-algebras. If $\phi[x_i] = \phi'[x_i]$ for any $1 \le i \le n$, then $\phi = \phi'$.*

Proof. First we show that the maps $\phi^*, \phi'^* : \mathrm{Spec}_r\, B \longrightarrow \mathrm{Spec}_r\, A$ coincide. Given a point $q \in \mathrm{Spec}_r\, B$, let us compare the cartesian coordinates of the points $p = \phi^*(q)$, $p' = \phi'^*(q)$ (recall that $\mathrm{Spec}_r\, A = (\mathfrak{a})_0 \subseteq \mathbb{R}^n$),

$$x_i(p) = \delta_p(x_i) = (\delta_q \circ \phi)(x_i) = \delta_q(\phi(x_i)) = \delta_q(\phi'(x_i)) = \cdots = x_i(p') \ ,$$

hence $p = p'$.

Now we prove that $\phi = \phi'$. Given points $q \in \mathrm{Spec}_r\, B$, $p = \phi^*(q) = \phi'^*(q)$, the morphisms $\phi_r, \phi'_r \colon A/\mathfrak{m}_p^{r+1} \longrightarrow B/\mathfrak{m}_q^{r+1}$ coincide because $A/\mathfrak{m}_p^{r+1}$ is generated by $[x_1], \ldots, [x_n]$. Hence the morphisms $\widehat{\phi}, \widehat{\phi}' \colon \widehat{A}_p \to \widehat{B}_q$ also coincide. Then, for any $a \in A$, the elements $\phi(a), \phi'(a)$ have the same Taylor expansion at any point $q \in \mathrm{Spec}_r\, B$,

$$j_q(\phi(a)) = \widehat{\phi}(j_p a) = \widehat{\phi}'(j_p a) = j_q(\phi'(a)) \ .$$

By 2.17 we conclude that $\phi(a) = \phi'(a)$.

□

Corollary 2.20. *Let A be a differentiable algebra and let $a_1, \ldots, a_n \in A$. There exists a unique morphism $\varphi^* \colon \mathcal{C}^\infty(\mathbb{R}^n) \to A$ such that $\varphi^*(x_i) = a_i$, $1 \leq i \leq n$.*

Proof. The uniqueness follows from 2.19. To show the existence, we may assume that $A = \mathcal{C}^\infty(\mathbb{R}^m)$. Then the differentiable map $\varphi = (a_1, \ldots, a_n) \colon \mathbb{R}^m \to \mathbb{R}^n$ induces a morphism $\varphi^* \colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^m)$ such that $\varphi^*(x_i) = a_i$.

□

Remark. Given $a_1, \ldots, a_n \in A$ and a function $h(x_1, \ldots, x_n) \in \mathcal{C}^\infty(\mathbb{R}^n)$, we may define $h(a_1, \ldots, a_n) := \varphi^*(h) \in A$, where φ^* is the morphism stated above. In other words, any differentiable algebra has a canonical structure of $\mathcal{C}^\infty$-ring (see [30] for definitions).

Corollary 2.21. *Let $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ and $B = \mathcal{C}^\infty(\mathbb{R}^m)/\mathfrak{b}$ be differentiable algebras. For any morphism $\phi : A \to B$ there exists a commutative square*

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^n) & \xrightarrow{\ \pi\ } & A \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} \\ \mathcal{C}^\infty(\mathbb{R}^m) & \xrightarrow{\ \pi\ } & B \end{array}$$

where π stands for the respective canonical projections.

Proof. If $\phi[x_i] = [f_i] \in B = \mathcal{C}^\infty(\mathbb{R}^m)/\mathfrak{b}$, $1 \leq i \leq n$, let us consider a morphism ψ such that $\psi(x_i) = f_i$. Then the square is commutative by the uniqueness stated in 2.19.

□

Corollary 2.22. *Let $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ and $B = \mathcal{C}^\infty(\mathbb{R}^m)/\mathfrak{b}$ be differentiable algebras endowed with the quotient topology. Any morphism of $\mathbb{R}$-algebras $\phi \colon A \to B$ is continuous.*

Proof. Since A is endowed with the quotient topology, we may assume that $\mathfrak{a} = 0$, i.e., $A = \mathcal{C}^\infty(\mathbb{R}^n)$. Let $y_1, \ldots, y_n \in \mathcal{C}^\infty(\mathbb{R}^m)$ such that $\phi(x_i) = [y_i]$. Let us consider the differentiable map $\varphi = (y_1, \ldots, y_n) \colon \mathbb{R}^m \to \mathbb{R}^n$ and the induced continuous morphism $\varphi^* \colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^m)$, $\varphi^*(f) = f \circ \varphi$. Note that $\varphi^*(x_i) = y_i$, so that ϕ coincides with the continuous composition map

$$\mathcal{C}^\infty(\mathbb{R}^n) \xrightarrow{\varphi^*} \mathcal{C}^\infty(\mathbb{R}^m) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^m)/\mathfrak{b} = B\ .$$

□

Theorem 2.23. *Any differentiable algebra A has a unique Fréchet topology, named* ***canonical topology****, such that any presentation $A \simeq \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ is a homeomorphism. Every morphism of differentiable algebras is continuous with respect to their canonical topologies.*

Proof. Given a presentation $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$, the quotient topology on A is Fréchet (quotients of Fréchet spaces with respect to closed subspaces are Fréchet). The uniqueness is a direct consequence of 2.22.

□

Corollary 2.24. *The kernel of any morphism of differentiable algebras $A \to B$ is a closed ideal of A.*

Proof. Any morphism $A \to B$ is continuous and the topology of B is separated.

□

Corollary 2.25. *An ideal $\mathfrak{a}$ of a differentiable algebra A is closed if and only if $A/\mathfrak{a}$ is a differentiable algebra.*

Proof. Let us consider an epimorphism $\pi \colon \mathcal{C}^\infty(\mathbb{R}^n) \to A$ (which is continuous by 2.23). If $\mathfrak{a}$ is closed in A, then $\mathfrak{b} = \pi^{-1}(\mathfrak{a})$ is a closed ideal in $\mathcal{C}^\infty(\mathbb{R}^n)$. Since $A/\mathfrak{a} \simeq \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{b}$, we conclude that $A/\mathfrak{a}$ is a differentiable algebra.

Conversely, if $A/\mathfrak{a}$ is a differentiable algebra, then the kernel $\mathfrak{a}$ of the canonical projection $A \to A/\mathfrak{a}$ is closed by 2.24.

□

Proposition 2.26. *Let A be a differentiable algebra and let $p \in \operatorname{Spec}_r A$. The quotient $A/\mathfrak{m}_p^{r+1}$ is a differentiable algebra for any $r \geq 0$. Hence $\mathfrak{m}_p^{r+1}$ is a closed ideal of A.*

Proof. Let us consider a presentation $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a} = A$. Then, by 2.13, $\operatorname{Spec}_r A = (\mathfrak{a})_0 \subseteq \mathbb{R}^n$, and we have an epimorphism

$$\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{m}_{\mathbb{R}^n,p}^{r+1} \ \longrightarrow\ A/\mathfrak{m}_p^{r+1}\ .$$

Now $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{m}_{\mathbb{R}^n,p}^{r+1}$ is a differentiable algebra since $\mathfrak{m}_{\mathbb{R}^n,p}^{r+1}$ is a closed ideal (2.11). Moreover, it is finite dimensional as a vector space, hence its ideals are closed. In particular, the kernel of the above epimorphism is closed and then 2.25 says that $A/\mathfrak{m}_p^{r+1}$ is a differentiable algebra.

□

Borel's theorem and Whitney's spectral theorem are easily generalized for differentiable algebras:

Theorem 2.27 (Borel's theorem). *Let A be a differentiable algebra. Given a point $p \in \operatorname{Spec}_r A$, the ideal $W_p = \{a \in A : j_p a = 0\}$ is closed and there exists an isomorphism*

$$A/W_p = \widehat{A}_p.$$

Therefore $\widehat{A}_p$ is a differentiable algebra.

Proof. Let us consider a presentation $\pi\colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a} = A$. By 2.16, we have an exact sequence

$$0 \longrightarrow \widehat{\mathfrak{a}}_p \longrightarrow \widehat{\mathcal{O}}_p \xrightarrow{\widehat{\pi}} \widehat{A}_p \longrightarrow 0\ .$$

In particular, $\widehat{\pi}\colon \widehat{\mathcal{O}}_p \to \widehat{A}_p$ is surjective. By the classical Borel's result, $j_p\colon \mathcal{C}^\infty(\mathbb{R}^n) \to \widehat{\mathcal{O}}_p$ also is surjective and then the commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^n) & \xrightarrow{\pi} & A \\ \big\downarrow{\scriptstyle j_p} & & \big\downarrow{\scriptstyle j_p} \\ \widehat{\mathcal{O}}_p & \xrightarrow{\widehat{\pi}} & \widehat{A}_p \end{array}$$

shows that $j_p\colon A \to \widehat{A}_p$ is surjective. Its kernel is W_p, so that $A/W_p = \widehat{A}_p$.

Finally, from 2.26 we obtain that $W_p = \bigcap_{r\in\mathbb{N}} \mathfrak{m}_p^{r+1}$ is a closed ideal of A. Then 2.25 shows that $A/W_p = \widehat{A}_p$ is a differentiable algebra.

□

Note that if $a = \sum_n a_n$ is a convergent series in A, then $j_x a = \sum_n j_x a_n$ is a convergent series in $\widehat{A}_x$, since the map $j_x\colon A \to \widehat{A}_x$ is continuous.

Theorem 2.28 (Spectral theorem). *Let A be a differentiable algebra. The closure of any ideal $\mathfrak{b} \subseteq A$ is the ideal*

$$\overline{\mathfrak{b}} \ = \ \{a \in A : j_p a \in j_p(\mathfrak{b}) \textit{ for any } p \in \operatorname{Spec}_r A\}\ .$$

Proof. Let us consider again a presentation $\pi\colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a} = A$. Since A is endowed with the quotient topology, it is clear that

$$\pi^{-1}(\overline{\mathfrak{b}}) = \overline{\pi^{-1}\mathfrak{b}}.$$

By Whitney's spectral theorem for $\mathcal{C}^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} \pi^{-1}(\overline{\mathfrak{b}}) = \overline{\pi^{-1}\mathfrak{b}} &= \{f \in \mathcal{C}^\infty(\mathbb{R}^n)\colon j_p f \in j_p(\pi^{-1}\mathfrak{b}) \text{ for any } p \in \mathbb{R}^n\} \\ &= \{f \in \mathcal{C}^\infty(\mathbb{R}^n)\colon j_p f \in j_p(\pi^{-1}\mathfrak{b}) \text{ for any } p \in \operatorname{Spec}_r A\} \end{aligned}$$

where the last equality is a consequence of the following facts: $j_p(\pi^{-1}\mathfrak{b}) = \widehat{\mathcal{O}}_p$ for any $p \notin (\pi^{-1}\mathfrak{b})_0$ and $(\pi^{-1}\mathfrak{b})_0 \subseteq (\mathfrak{a})_0 = \operatorname{Spec}_r A$.

Now, using the commutative diagram of epimorphisms

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^n) & \xrightarrow{\ \pi\ } & A \\ \Big\downarrow{\scriptstyle j_p} & & \Big\downarrow{\scriptstyle j_p} \\ \widehat{\mathcal{O}}_p & \xrightarrow{\ \widehat{\pi}\ } & \widehat{A}_p \end{array}$$

and applying π in the equality

$$\pi^{-1}(\overline{\mathfrak{b}}) \ = \ \{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \ j_p f \in j_p(\pi^{-1}\mathfrak{b}) \text{ for any } p \in \operatorname{Spec}_r A\}$$

we obtain the desired result.

□

Note 2.29. Let A a differentiable algebra. There is a spectral theorem for modules (see [35] II.4.8) determining closed submodules of A^n. The following reformulation may be useful: A finitely generated A-module M admits a Fréchet A-module structure (see section 6.1 for definitions) if and only if

$$\bigcap_{x,r} \mathfrak{m}_x^r M \ = \ 0 \qquad\qquad (x \in \operatorname{Spec}_r A, \ r \in \mathbb{N})$$

(that is to say, an element $m \in M$ is zero if $j_x^r m := [m] = 0$ in $M/\mathfrak{m}_x^r M$ for any $x \in \operatorname{Spec}_r A, \ r \in \mathbb{N}$).

2.5 Examples

Definition. We say that an $\mathbb{R}$-algebra A is **reduced** when 0 is the unique element of A vanishing at any point of $\operatorname{Spec}_r A$. That is to say, when the spectral representation morphism $A \to \mathcal{C}(\operatorname{Spec}_r A, \mathbb{R}), \ f \mapsto \hat{f}$, defined in section 2.1 is injective, so that any reduced $\mathbb{R}$-algebra is canonically isomorphic to an algebra of real valued continuous functions on the topological space $\operatorname{Spec}_r A$.

Reduced algebras are also named **semisimple** in the literature. According to a theorem of Michael [29] any two structures of Fréchet algebra on a semisimple algebra coincide; hence the canonical topology of a reduced differentiable algebra A is the unique Fréchet algebra structure on A.

Example 2.30. Given a closed set Y in $\mathbb{R}^n$, let A_Y be the algebra of all continuous functions on Y which are restriction of a differentiable function on $\mathbb{R}^n$. It is a differentiable algebra because

$$\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{p}_Y \ \simeq \ A_Y \quad , \quad [f] \mapsto f|_Y \ ,$$

where $\mathfrak{p}_Y$ denotes the ideal of all differentiable functions vanishing on Y, which is a closed ideal:

$$\mathfrak{p}_Y = \bigcap_{y\in Y} \mathfrak{m}_y \ .$$

Since differentiable functions on $\mathbb{R}^n$ separate disjoint closed sets, we have $Y = (\mathfrak{p}_Y)_0 = \mathrm{Spec}_r\, A_Y$. In particular A_Y is a reduced differentiable algebra.

Let us consider the particular case when Y is a closed smooth submanifold of $\mathbb{R}^n$. By 2.4 the restriction morphism $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(Y)$ is surjective; hence $\mathcal{C}^\infty(Y) = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{p}_Y = A_Y$ is a reduced differentiable algebra. In this case, the canonical topology of $\mathcal{C}^\infty(Y)$ is just the usual Fréchet topology: The surjective map $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(Y)$ is continuous for the usual topologies, so that the usual topology is weaker than the canonical topology; hence they coincide since both are Fréchet.

Using Whitney's embedding theorem, we obtain the following fact: *If $\mathcal{V}$ is a separated smooth manifold of bounded dimension whose topology has a countable basis, then $\mathcal{C}^\infty(\mathcal{V})$ is a differentiable algebra.*

Example 2.31. Let A be a differentiable algebra and let Y be a closed set in $\mathrm{Spec}_r\, A$. Each element $f \in A$ defines a continuous function $\hat{f}|_Y : Y \to \mathbb{R}$, and the algebra B of all continuous functions on Y so obtained is a reduced differentiable algebra and $\mathrm{Spec}_r\, B = Y$. In fact, we have $B \simeq A/\mathfrak{p}_Y$ where $\mathfrak{p}_Y = \bigcap_{y\in Y} \mathfrak{m}_y$. By 2.24, each ideal $\mathfrak{m}_y$ is closed in A, hence $\mathfrak{p}_Y$ also is closed and then $B = A/\mathfrak{p}_Y$ is a differentiable algebra by 2.25. Since any closed set in $\mathrm{Spec}_r\, A$ is a zero-set (2.14), we have $Y = (\mathfrak{p}_Y)_0$. By 2.12 we conclude that $Y = \mathrm{Spec}_r\, B$.

When $Y = \mathrm{Spec}_r\, A$, we obtain a reduced differentiable algebra $A_{red} := A/\mathfrak{r}_A$, where $\mathfrak{r}_A = \{f \in A\colon\ \hat{f} = 0\}$. By definition A is reduced if and only if $\mathfrak{r}_A = 0$; that is to say, when the canonical projection $\pi\colon A \to A_{red}$ is an isomorphism. If $j\colon A \to B$ is a morphism of differentiable algebras, it is clear that $j(\mathfrak{r}_A) \subseteq \mathfrak{r}_B$. Therefore j induces a morphism $j_{red}\colon A_{red} \to B_{red}$ such that the following square is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\ j\ } & B \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ A_{red} & \xrightarrow{\ j_{red}\ } & B_{red} \end{array}$$

In particular, if B is reduced, any morphism $A \to B$ factors through the canonical projection $\pi\colon A \to A_{red}$, i.e., we have a bijection

$$\mathrm{Hom}_{\mathbb{R}\text{-alg}}(A_{red}, B) \ = \ \mathrm{Hom}_{\mathbb{R}\text{-alg}}(A, B) \quad , \quad h \mapsto h\pi \ .$$

Example 2.32. Finite direct products of differentiable algebras are differentiable algebras.

In fact, let $A_1 = \mathcal{C}^\infty(\mathbb{R}^{n_1})/\mathfrak{a}_1$, $A_2 = \mathcal{C}^\infty(\mathbb{R}^{n_2})/\mathfrak{a}_2$ be differentiable algebras. The disjoint union $\mathbb{R}^{n_1} \coprod \mathbb{R}^{n_2}$ is a closed smooth submanifold of some $\mathbb{R}^n$ (take $n > n_1$ and $n > n_2$); hence, according to example 2.30,

$$\mathcal{C}^\infty(\mathbb{R}^{n_1} \textstyle\coprod \mathbb{R}^{n_2}) \ = \ \mathcal{C}^\infty(\mathbb{R}^{n_1}) \oplus \mathcal{C}^\infty(\mathbb{R}^{n_2})$$

is a differentiable algebra. Now the ideal $\mathfrak{a}_1 \oplus \mathfrak{a}_2$ is closed in $\mathcal{C}^\infty(\mathbb{R}^{n_1}) \oplus \mathcal{C}^\infty(\mathbb{R}^{n_2})$ and we conclude that

$$A_1 \oplus A_2 \;=\; \mathcal{C}^\infty(\mathbb{R}^{n_1})/\mathfrak{a}_1 \oplus \mathcal{C}^\infty(\mathbb{R}^{n_2})/\mathfrak{a}_2 \;=\; (\mathcal{C}^\infty(\mathbb{R}^{n_1}) \oplus \mathcal{C}^\infty(\mathbb{R}^{n_2}))/(\mathfrak{a}_1 \oplus \mathfrak{a}_2)$$

is a differentiable algebra. Moreover, $A_1 \oplus A_2$ is reduced when so are the algebras A_i, since it is easy to check that

$$\operatorname{Spec}_r(A_1 \oplus A_2) \;=\; (\operatorname{Spec}_r A_1) \coprod (\operatorname{Spec}_r A_2)\ .$$

Example 2.33. The algebra $\mathbb{R}[\varepsilon] := \mathbb{R} \oplus \mathbb{R}\,\varepsilon$, where $\varepsilon^2 = 0$, is a differentiable algebra because

$$\mathcal{C}^\infty(\mathbb{R})/\mathfrak{m}_0^2 \;\simeq\; \mathbb{R}[\varepsilon] \quad , \quad [f(t)] \mapsto f(0) + f'(0)\varepsilon\ ,$$

and $\mathfrak{m}_0^2 = \{f \in \mathcal{C}^\infty(\mathbb{R})\colon\ f(0) = f'(0) = 0\}$ is a closed ideal. The real spectrum of this algebra has a unique point: $\operatorname{Spec}_r \mathbb{R}[\varepsilon] = (\mathfrak{m}_0^2)_0 = \{0\} \subset \mathbb{R}$. It is clear that $\mathbb{R}[\varepsilon]$ is not reduced.

Example 2.34. Let Y be a closed set in $\mathbb{R}^n$. The Whitney ideal

$$W_Y \;:=\; \{f \in \mathcal{C}^\infty(\mathbb{R}^n)\colon\ j_y f = 0, \forall y \in Y\} \;=\; \bigcap_{y,r} \mathfrak{m}_y^r \qquad (y \in Y,\ r \in \mathbb{N})$$

is closed, and the differentiable algebra $\mathcal{C}^\infty(\mathbb{R}^n)/W_Y$ is called the Whitney algebra of Y in $\mathbb{R}^n$. In general, if A is a differentiable algebra and Y is a closed set in $X = \operatorname{Spec}_r A$, then the **Whitney ideal** of Y in X

$$W_{Y/X} \;:=\; \{a \in A : j_y a = 0, \forall y \in Y\} \;=\; \bigcap_{y,r} \mathfrak{m}_y^r \qquad (y \in Y,\ r \in \mathbb{N})$$

is a closed ideal of A and we say that the differentiable algebra $A/W_{Y/X}$ is the **Whitney algebra** of Y in X.

Example 2.35. If A is a differentiable algebra and $f \in A$, then f^2+1 is invertible because it does not vanish at any point of $\operatorname{Spec}_r A$. Therefore, if an $\mathbb{R}$-algebra A has some maximal ideal $\mathfrak{m}$ with residue field $A/\mathfrak{m} \simeq \mathbb{C}$, then A is not a differentiable algebra. In particular, $\mathbb{C}$ is not a differentiable algebra.

Example 2.36. Rings of germs of differentiable functions are *not* differentiable algebras, except in some very trivial cases. In fact, the ring $\mathcal{O}_{\mathbb{R}^n,x}$ of germs of $\mathcal{C}^\infty$-functions at a point $x \in \mathbb{R}^n$ is just

$$\mathcal{O}_{\mathbb{R}^n,x} \;=\; \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{n}_x\ ,$$

where $\mathfrak{n}_x$ is the ideal of all functions vanishing on a neighbourhood of x (see 1.6). By Whitney's spectral theorem, its closure $\overline{\mathfrak{n}}_x$ is just the Whitney ideal W_x of all flat functions at x, so that $\mathfrak{n}_x$ is not a closed ideal when $n \geq 1$. We may conclude that $\mathcal{O}_{\mathbb{R}^n,x}$ is not a differentiable algebra by 2.24.

By the way, there exist morphisms of $\mathbb{R}$-algebras $\varphi^*\colon \mathcal{O}_{\mathbb{R}^n,x} \to \mathcal{O}_{\mathbb{R}^n,x}$ which are not induced by a differentiable map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ (see [47]).

Example 2.37. Let A be a finite $\mathbb{R}$-algebra (= of finite dimension as a real vector space). Then A is a differentiable algebra if and only if A is rational (i.e., every maximal ideal of A is a real ideal). See theorem 9.2

3 Differentiable Spaces

In this chapter we introduce differentiable spaces as ringed spaces locally isomorphic to the real spectrum of some differentiable algebra. We start by defining the structural sheaf of rings $\widetilde{A}$ on $\operatorname{Spec}_r A$ for any differentiable algebra A. The main result is the Localization theorem for differentiable algebras [37, 42], which is the basis for the algebraic point of view adopted in these notes.

3.1 Localization of Differentiable Algebras

Let A be an $\mathbb{R}$-algebra and let U be an open set in $\operatorname{Spec}_r A$. We shall denote by A_U the localization (ring of fractions, [1] Chapter 3) of A with respect to the multiplicative set of all elements in A without zeroes in U. In other words, elements in A_U are (equivalence classes of) fractions $\frac{a}{s}$, where $a, s \in A$ and $s(x) \neq 0$ for any $x \in U$.

We have a canonical morphism $A \to A_U$, $a \mapsto \frac{a}{1}$, with the following universal property: Any morphism of $\mathbb{R}$-algebras $\phi\colon A \to B$, such that $\phi(s)$ is invertible in B for any $s \in A$ without zeroes in U, factors through the morphism $A \to A_U$.

Lemma 3.1. *Let A be a differentiable algebra. For any closed set Y in* $\operatorname{Spec}_r A$ *there exists $s \in A$ such that $Y = (s)_0$.*

Proof. We may put $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ and $Y \subseteq \operatorname{Spec}_r A = (\mathfrak{a})_0 \subseteq \mathbb{R}^n$. By 2.10 there exists $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $Y = (f)_0$. Then $s := [f] \in \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ is the desired element.

□

Proposition 3.2. *Let A be a differentiable algebra. For any open set U in* $\operatorname{Spec}_r A$ *we have a homeomorphism*

$$\operatorname{Spec}_r A_U = U.$$

Proof. Let $S = \{s \in A : s(x) \neq 0, \forall x \in U\}$. Using the universal property of the localization and that U is a cozero set (3.1), we obtain the following bijections

$$\operatorname{Spec}_r A_U = \operatorname{Hom}_{\mathbb{R}\text{-alg}}(A_U, \mathbb{R}) = \{\delta_x\colon A \to \mathbb{R} : \delta_x(s) \neq 0, \forall s \in S\}$$
$$= \{x \in \operatorname{Spec}_r A : s(x) \neq 0, \forall s \in S\} = U.$$

Finally, it is easy to check that the above bijections are homeomorphisms.

□

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Definition. Let A be an $\mathbb{R}$-algebra and let M be an A-module. If U is an open set in $\operatorname{Spec}_r A$, then M_U will denote the localization of M with respect to the multiplicative set of all elements $s \in A$ which does not vanish at any point of U. Therefore elements in M_U are (equivalent classes of) fractions m/s where $m \in M$ and $s \in A$ without zeroes in U.

The sheaf on $\operatorname{Spec}_r A$ associated to the presheaf $U \rightsquigarrow M_U$ will be denoted by $\widetilde{M}$, and we say that it is the **sheaf associated** to M. The sheaf of rings $\widetilde{A}$ associated to the presheaf $U \rightsquigarrow A_U$ is called the **structural sheaf** on $\operatorname{Spec}_r A$.

Note that $\widetilde{M}$ is an $\widetilde{A}$-module and that any morphism of A-modules $M \to N$ induces a morphism of $\widetilde{A}$-modules $\widetilde{M} \to \widetilde{N}$. Moreover, the functor $M \rightsquigarrow \widetilde{M}$ transforms exact sequences of A-modules into exact sequences of $\widetilde{A}$-modules, because the functor $M \rightsquigarrow M_U$ is exact ([1] 3.3).

Given a point $x \in \operatorname{Spec}_r A$, we shall denote by M_x the localization of M with respect to the multiplicative system $\{s \in A : s(x) \neq 0\}$. Therefore, elements in M_x are (equivalent classes of) fractions m/s where $m \in M$, $s \in A$ and $s(x) \neq 0$. Note that the stalk $(\widetilde{M})_x$ of $\widetilde{M}$ at any point $x \in \operatorname{Spec}_r A$ coincides with M_x.

Any element $m \in M$ defines a global section of $\widetilde{M}$ and its germ at a point $x \in \operatorname{Spec}_r A$ is denoted by m_x, i.e., $m_x = \frac{m}{1} \in M_x$. The **support** of m is defined to be its support as a section of $\widetilde{M}$:

$$\operatorname{Supp}(m) := \{x \in \operatorname{Spec}_r A : m_x \neq 0\} .$$

Example 3.3. When $A = \mathcal{C}^\infty(\mathbb{R}^n)$, we have $\mathbb{R}^n = \operatorname{Spec}_r A$ by 2.1, and the Localization theorem for differentiable functions shows that $A_U = \mathcal{C}^\infty(U)$; hence the presheaf $U \rightsquigarrow A_U$ is a sheaf and $\widetilde{A}$ coincides with the sheaf $\mathcal{C}^\infty_{\mathbb{R}^n}$ of differentiable functions on $\mathbb{R}^n$.

Proposition 3.4. *Let A be a differentiable algebra and let $x \in \operatorname{Spec}_r A$. The natural morphism $A \to (\widetilde{A})_x = A_x$ is surjective.*

Proof. A is a quotient of $\mathcal{C}^\infty(\mathbb{R}^n)$, the localization functor is exact, and the result holds for $\mathcal{C}^\infty(\mathbb{R}^n)$ by 1.6.

□

Lemma 3.5. *If U is an open set in $\mathbb{R}^n$, then $\mathcal{C}^\infty(U)$ is a differentiable algebra and its canonical topology is the usual Fréchet topology.*

Proof. Let f be a differentiable function on $\mathbb{R}^n$ such that $(f)_0 = \mathbb{R}^n - U$ (it exists by 2.10). The graph $i\colon U \hookrightarrow \mathbb{R}^{n+1}$ of $1/f$ is a closed embedding. By 2.4, the continuous morphism $i^*\colon \mathcal{C}^\infty(\mathbb{R}^{n+1}) \to \mathcal{C}^\infty(U)$ (with respect to the usual topologies) is surjective. Its kernel is a closed ideal and we conclude that $\mathcal{C}^\infty(U)$ is a differentiable algebra. Finally, the canonical topology of $\mathcal{C}^\infty(U)$ is finer than the usual topology, and they coincide since both are Fréchet.

□

Lemma 3.6. *Let $\mathfrak{a}$ be a closed ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$ and let U be an open set in $\mathbb{R}^n$. Then $\mathfrak{a}_U$ is a closed ideal of $\mathcal{C}^\infty(U)$ and $\widetilde{\mathfrak{a}}\,(U) = \mathfrak{a}_U$.*

Proof. We use the notations of the proof of the Localization theorem for differentiable functions. Given $f \in \mathcal{C}^\infty(U)$ we have $f = g/h$, where

$$g = \sum_{i=1}^{\infty} 2^{-i} \frac{g_i f}{1 + q_i(g_i f) + q_i(g_i)} \ ,$$
$$h = \sum_{i=1}^{\infty} 2^{-i} \frac{g_i}{1 + q_i(g_i f) + q_i(g_i)} \ .$$

Now, if $j_x f \in \widehat{\mathfrak{a}}_x$ at any point $x \in U$, then $j_x(g_n f) \in \widehat{\mathfrak{a}}_x$ at any point $x \in \mathbb{R}^n$ because g_n vanishes in a neighbourhood of $\mathbb{R}^n - U$. By Whitney's spectral theorem, we have $g_n f \in \mathfrak{a}$ and, the ideal $\mathfrak{a}$ being closed, we obtain that $g \in \mathfrak{a}$ and $f = g/h \in \mathfrak{a}_U$. That is to say

$$\mathfrak{a}_U = \{f \in \mathcal{C}^\infty(U) \colon \ j_x f \in \widehat{\mathfrak{a}}_x \text{ for any } x \in U\}$$

and we conclude that $\mathfrak{a}_U$ is a closed ideal of $\mathcal{C}^\infty(U)$ and that $\tilde{\mathfrak{a}} \colon U \rightsquigarrow \mathfrak{a}_U$ is a sheaf of ideals.

□

Theorem 3.7. *Let A be a differentiable algebra and let U be an open subset of* $\text{Spec}_r\, A$*. Then A_U is a differentiable algebra.*

Proof. Let $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ and then $\text{Spec}_r A = (\mathfrak{a})_0 \subseteq \mathbb{R}^n$. Let U' be an open set in $\mathbb{R}^n$ such that $U = U' \cap (\mathfrak{a})_0$. By 3.5, 3.6, and the Localization theorem for differentiable functions, we obtain that

$$A_{U'} \ = \ \mathcal{C}^\infty(\mathbb{R}^n)_{U'}/\mathfrak{a}_{U'} \ = \ \mathcal{C}^\infty(U')/\mathfrak{a}_{U'}$$

is a differentiable algebra.

Now, we have to prove that $A_{U'} = A_U$. It is easy to check that $\text{Spec}_r\, A_{U'} = U' \cap (\mathfrak{a})_0 = U$. By 2.15, if $s \in A$ does not vanish at any point of U, then s is invertible in $A_{U'}$; hence the natural morphism $A_{U'} \to A_U$ is an isomorphism.

□

Lemma 3.8. *Let A be a differentiable algebra and let $X = \text{Spec}_r A$. Given an open subset $U \subseteq X$ and a compact subset $K \subseteq U$, there exists $a \in A$ such that $a \geq 0$ on X, $a_x = 1$ for any $x \in K$ and* $\text{Supp}\,(a) \subseteq U$.

Proof. Let $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ and then $X = (\mathfrak{a})_0 \subseteq \mathbb{R}^n$. Let $U' \subseteq \mathbb{R}^n$ be an open subset such that $U = U' \cap X$. Let $f \geq 0$ be a differentiable function on $\mathbb{R}^n$ such that $f_x = 1$ for any $x \in K$ and $\text{Supp}\,(f) \subseteq U'$. Then $a := [f]$ is the desired element.

□

Let $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ be a differentiable algebra. Given two disjoint closed sets X, Y in $\text{Spec}_r A \subseteq \mathbb{R}^n$, according to 1.7, some element of A defines a global section a of $\widetilde{A}$ such that $a = 1$ on a neighbourhood of X and $a = 0$ on a neighbourhood of Y. Therefore, *the sheaf of rings $\widetilde{A}$ is soft, so that any $\widetilde{A}$-module is also a soft sheaf* ([15], II 3.7.1, 3.7.2).

Lemma 3.9. *Let $a = \sum a_n$ be a convergent series in a differentiable algebra A and let $x \in \mathrm{Spec}_r A$. If there exists a neighbourhood V of x which intersects only a finite number of members of the family $\{\mathrm{Supp}\,(a_n)\}$, then $a_x = \sum_n (a_n)_x$.*

Proof. Let $b \in A$ such that $b_x = 1$ and $\mathrm{Supp}\,(b) \subseteq V$. Then $ab = \sum (a_n b)$ only has a finite number of non-zero summands (note that if $\mathrm{Supp}\,(a_n b) = \emptyset$ then $a_n b = 0$ by 2.17), hence

$$a_x = (ab)_x = \Big(\sum_n a_n b\Big)_x = \sum_n (a_n b)_x = \sum_n (a_n)_x \ .$$

□

Localization theorem for differentiable algebras ([41]). *Let A be a differentiable algebra and let $\tilde{A}$ be the structural sheaf on $X = \mathrm{Spec}_r\, A$. For any open set U in $\mathrm{Spec}_r\, A$ we have*

$$\tilde{A}\,(U) = A_U \ .$$

In particular, $\tilde{A}\,(X) = A$.

Moreover, $\tilde{A}\,(U)$ is a differentiable algebra and $U = \mathrm{Spec}_r\, \tilde{A}\,(U)$.

Proof. (1) The natural morphism $A_U \to \tilde{A}\,(U)$ is injective:

If $a \in A_U$ vanishes as a section of $\tilde{A}\,(U)$, then $a_x = 0$ in $(\tilde{A})_x = A_x$ for any $x \in U$. Since A_U is a differentiable algebra, we obtain that $a = 0$ by 2.17.

(2) The natural morphism $A_U \to \tilde{A}\,(U)$ is surjective:

Let $\tilde{a} \in \tilde{A}\,(U)$. Since the morphism $A \to (\tilde{A})_x = A_x$ is surjective (3.4), for any point $x \in U$ there exists by 3.8 an element $b \in A$ such that $b(x) \neq 0$, $b \geq 0$ on X, $\mathrm{Supp}\,(b) \subseteq U$, and $b\tilde{a}$ is a global section which is in A. We may choose a sequence $\{b_i\}$ of such elements, so that $\{\mathrm{Supp}\,(b_i)\}$ is a locally finite family whose interior sets cover U. Let $q_1 \leq q_2 \leq \dots$ be seminorms defining the topology of A. The convergent series

$$a = \sum_{i=1}^{\infty} 2^{-i} \frac{b_i \tilde{a}}{1 + q_i(b_i \tilde{a}) + q_i(b_i)} \ ,$$

$$s = \sum_{i=1}^{\infty} 2^{-i} \frac{b_i}{1 + q_i(b_i \tilde{a}) + q_i(b_i)} \ .$$

define elements $a, s \in A$, where s does not vanish at any point of U, and $\tilde{a} = a/s$ because both sections have the same germ at any point of U by 3.9.

(3) $\tilde{A}\,(X) = A_X = A$ by 2.15. Finally, $\tilde{A}\,(U)$ is a differentiable algebra by 3.7 and $U = \mathrm{Spec}_r\, \tilde{A}\,(U)$ by 3.2.

□

Remark 3.10. The same argument proves the following result: *Let A be a differentiable algebra and let U be an open subset of $X = \mathrm{Spec}_r\, A$. If I is a closed ideal of A, then $\tilde{I}\,(U) = I_U$. In particular, $\tilde{I}\,(X) = I$.*

Theorem 3.11. *Let A be a differentiable algebra and let $X = \operatorname{Spec}_r A$. Any $\widetilde{A}$-module $\mathcal{M}$ is fully determined by the A-module of global sections $M := \mathcal{M}(X)$ because*

$$\mathcal{M} = \widetilde{M} .$$

Therefore the functors $\mathcal{M} \rightsquigarrow \mathcal{M}(X)$ and $M \rightsquigarrow \widetilde{M}$ define an equivalence of the category of $\widetilde{A}$-modules with a full subcategory of A-modules (A-modules M such that the natural morphism $M \to \widetilde{M}(X)$ is an isomorphism). In particular:

$$\operatorname{Hom}_{\widetilde{A}}(\mathcal{M}, \mathcal{N}) = \operatorname{Hom}_A(\mathcal{M}(X), \mathcal{N}(X)) .$$

Moreover, if $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is an exact sequence of $\widetilde{A}$-modules, then the sequence of A-modules $0 \to \mathcal{M}'(X) \to \mathcal{M}(X) \to \mathcal{M}''(X) \to 0$ is exact.

Proof. First we prove that the natural morphisms $M_U \to \mathcal{M}(U)$ define an isomorphism of $\widetilde{A}$-modules $\widetilde{M} \to \mathcal{M}$:

Let $m \in \mathcal{M}(U)$ and let $x \in U$. There exists some $a \in A$ such that $a = 0$ outside a small neighbourhood of x and $a = 1$ on a neighbourhood of x, so that the germ $m_x = (am)_x$ coincides with the germ of a global section of $\mathcal{M}$. Therefore, the natural morphism $M \to \mathcal{M}_x$ is surjective, hence so is $M_x \to \mathcal{M}_x$.

When $U = \operatorname{Spec}_r A$ and $m_x = 0$, we may choose a so that $am = 0$. Since $a \in A - \mathfrak{m}_x$, we conclude that $m = 0$ in M_x, i.e., the morphism $M_x \to \mathcal{M}_x$ is injective.

Finally, the functor of global sections is exact because any $\widetilde{A}$-module $\mathcal{M}'$ is a soft sheaf and then $H^1(X, \mathcal{M}') = 0$ ([15] II 4.4.3).

□

Corollary 3.12. *Let A be a differentiable algebra and let $X = \operatorname{Spec}_r A$. If M is an A-module of finite presentation, then $M = \widetilde{M}(X)$.*

Proof. Any presentation $A^m \to A^n \to M \to 0$ induces an exact sequence of sheaves $(\widetilde{A})^m \to (\widetilde{A})^n \to \widetilde{M} \to 0$. According to 3.11, we have an exact sequence of A-modules

$$\widetilde{A}(X)^m \longrightarrow \widetilde{A}(X)^n \longrightarrow \widetilde{M}(X) \longrightarrow 0$$

and we may conclude because $\widetilde{A}(X) = A$.

□

Corollary 3.13. *Let A be a differentiable algebra and let $X = \operatorname{Spec}_r A$. If I is a finitely generated ideal of A, then $I = \widetilde{I}(X)$.*

Proof. Since A/I is an A-module of finite presentation, taking global sections in the exact sequence of sheaves

$$0 \longrightarrow \widetilde{I} \longrightarrow \widetilde{A} \longrightarrow (A/I)^{\sim} \longrightarrow 0 ,$$

we obtain the following exact sequence of A-modules:

$$0 \longrightarrow \widetilde{I}(X) \longrightarrow A \longrightarrow A/I \longrightarrow 0 .$$

□

3.2 Differentiable Spaces

Definition. A **locally ringed $\mathbb{R}$-space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of $\mathbb{R}$-algebras on X such that the stalk $\mathcal{O}_{X,x}$ at any point $x \in X$ is a **local** ring (with a unique maximal ideal that we denote by $\mathfrak{m}_{X,x}$ or $\mathfrak{m}_x$).

If $f \in \mathcal{O}_X(U)$ and $x \in U$, then the residue class of the germ of f in $\mathcal{O}_{X,x}/\mathfrak{m}_x$ will be denoted by $f(x)$ and it is said to be the **value** of f at x. The condition of $\mathcal{O}_{X,x}$ being local means that f is invertible in some neighbourhood of x whenever $f(x) \neq 0$.

A **morphism** $(\varphi, \varphi^*) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ **of locally ringed $\mathbb{R}$-spaces** is a continuous map $\varphi : Y \to X$ and a morphism $\varphi^* : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ of sheaves of $\mathbb{R}$-algebras (or, equivalently, a morphism $\varphi^* : \varphi^* \mathcal{O}_X \to \mathcal{O}_Y$ of sheaves of $\mathbb{R}$-algebras) such that the morphisms $\varphi^*_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$, where $\varphi(y) = x$, are **local**, in the sense that $\varphi^*_y(\mathfrak{m}_x) \subseteq \mathfrak{m}_y$. This condition means that

$$(\varphi^* f)(y) = 0 \Leftrightarrow f(\varphi y) = 0 \, .$$

When the natural morphisms $\mathbb{R} \to \mathcal{O}_{X,x}/\mathfrak{m}_x$ are isomorphisms, then the map $\mathbb{R} = \mathcal{O}_{X,x}/\mathfrak{m}_x \to \mathcal{O}_{Y,y}/\mathfrak{m}_y$ induced by φ^*_y is the structural morphism, so that the condition of $\varphi^*_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ being local means that

$$\boxed{(\varphi^* f)(y) \; = \; f(\varphi y)}$$

(but this formula does not fully determine $\varphi^* f$ in terms of φ and f, because $\varphi^* f$ is not a map $Y \to \mathbb{R}$).

A morphism $(\varphi, \varphi^*) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally ringed $\mathbb{R}$-spaces is said to be an **isomorphism** if $\varphi : Y \to X$ is a homeomorphism and $\varphi^* : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ is an isomorphism of sheaves.

Morphisms of locally ringed $\mathbb{R}$-spaces may be obviously composed. The set of all morphisms of locally ringed $\mathbb{R}$-spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ will be denoted by $\operatorname{Hom}(Y, X)$. If no confusion is possible, a morphism of locally ringed $\mathbb{R}$-spaces (φ, φ^*) is denoted by φ.

Definition. The **real spectrum** of a differentiable algebra A is defined to be the locally ringed $\mathbb{R}$-space $(\operatorname{Spec}_r A \, , \, \widetilde{A})$. When no confusion is possible, it is denoted by $\operatorname{Spec}_r A$.

Definition. A locally ringed $\mathbb{R}$-space $(X, \mathcal{O}_X)$ is said to be an **affine differentiable space** (of finite type) if it is isomorphic to the real spectrum of some differentiable algebra. By the Localization theorem, this differentiable algebra must be $\mathcal{O}_X(X)$.

Proposition 3.14. *Let $(X, \mathcal{O}_X)$ be an affine differentiable space. If U is an open set in X, then $(U, \mathcal{O}_X|_U)$ is an affine differentiable space.*

Proof. Let A be a differentiable algebra and let U be an open set in $\operatorname{Spec}_r A$. The Localization theorem for differentiable algebras says that A_U is a differentiable algebra, $\operatorname{Spec}_r A_U = U$ and $\widetilde{A}|_U = \widetilde{A_U}$; hence $(U, \widetilde{A}|_U)$ is an affine differentiable space.

□

Definition. A locally ringed $\mathbb{R}$-space $(X, \mathcal{O}_X)$ is said to be a **differentiable space** (locally of finite type) if any point $x \in X$ has an open neighbourhood U in X such that $(U, \mathcal{O}_X|_U)$ is an affine differentiable space. Such open subsets of X are said to be **affine open sets** and they define a basis for the topology of X according to 3.14. **Morphisms of differentiable spaces** are defined to be morphisms of locally ringed $\mathbb{R}$-spaces.

According to 3.14, if $(X, \mathcal{O}_X)$ is a differentiable space and U is an open set in X, then $(U, \mathcal{O}_X|_U)$ is a differentiable space, and we denote it by U.

Let $(X, \mathcal{O}_X)$ be a differentiable space. Sections of $\mathcal{O}_X$ on an open set $U \subseteq X$ are said to be **differentiable functions** on U. The germ of a differentiable function $f \in \mathcal{O}_X(U)$ at a point $x \in U$ is denoted by f_x, and its residue class $f(x)$ in $\mathcal{O}_{X,x}/\mathfrak{m}_x = \mathbb{R}$ is said to be the **value** of f at x. Hence, any differentiable function $f \in \mathcal{O}_X(U)$ defines a map $\hat{f}\colon U \to \mathbb{R}$, $\hat{f}(x) = f(x)$, which is continuous because so it is on any affine open set (but the function f is not determined by the map $\hat{f}$). In particular, the zero-set of f is a closed subset of U.

If a differentiable function $f \in \mathcal{O}_X(U)$ does not vanish at any point of U, then f is invertible in $\mathcal{O}_X(U)$ because the rings of germs $\mathcal{O}_{X,x}$ are local rings.

Example 3.15. Let $\mathcal{C}^\infty_{\mathbb{R}^n}$ be the sheaf of differentiable functions on $\mathbb{R}^n$. According to 3.3, $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ is isomorphic to the real spectrum of $\mathcal{C}^\infty(\mathbb{R}^n)$; hence it is an affine differentiable space and we denote it by $\mathbb{R}^n$:

$$\mathbb{R}^n \ := \ (\operatorname{Spec}_r \mathcal{C}^\infty(\mathbb{R}^n), \mathcal{C}^\infty_{\mathbb{R}^n}) \ .$$

Example 3.16. Let $(\mathcal{V}, \mathcal{C}^\infty_{\mathcal{V}})$ be a smooth manifold. By definition, any point $x \in \mathcal{V}$ has an open neighbourhood U such that $(U, \mathcal{C}^\infty_{\mathcal{V}}|_U)$ is isomorphic to an open set in $\mathbb{R}^n = (\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$; hence $(\mathcal{V}, \mathcal{C}^\infty_{\mathcal{V}})$ is a differentiable space. *Any smooth manifold is a differentiable space.* Affine smooth manifolds are just smooth manifolds which admit a closed embedding in some affine space $\mathbb{R}^n$.

If $\mathcal{V}$ and $\mathcal{W}$ are smooth manifolds, any differentiable map $\psi\colon \mathcal{V} \to \mathcal{W}$ defines a morphism of differentiable spaces $(\psi, \psi^*)\colon (\mathcal{V}, \mathcal{C}^\infty_{\mathcal{V}}) \to (\mathcal{W}, \mathcal{C}^\infty_{\mathcal{W}})$, where the morphisms $\psi^*\colon \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(\psi^{-1}U)$ are defined to be $\psi^*(f) := f \circ \psi$. So we get all the morphisms of differentiable spaces $(\mathcal{V}, \mathcal{C}^\infty_{\mathcal{V}}) \to (\mathcal{W}, \mathcal{C}^\infty_{\mathcal{W}})$. In fact, if $(\varphi, \varphi^*)\colon (\mathcal{V}, \mathcal{C}^\infty_{\mathcal{V}}) \to (\mathcal{W}, \mathcal{C}^\infty_{\mathcal{W}})$ is a morphism of differentiable spaces, then φ is a continuous map and we have $(\varphi^* f)(p) = f(\varphi(p))$ for any $p \in \mathcal{V}$. In particular $f \circ \varphi \in \mathcal{C}^\infty(\mathcal{V})$ for any $f \in \mathcal{C}^\infty(\mathcal{W})$, so that φ is a differentiable map by 2.2. *Morphisms of differentiable spaces between smooth manifolds are just differentiable maps.*

Example 3.17 ("Recollement" of differentiable spaces). Any locally ringed $\mathbb{R}$-space obtained by "recollement" of differentiable spaces ([17] 0.4.1.7) is a differentiable space.

Therefore, any disjoint union $\coprod_i X_i$ of a family $\{X_i\}_{i\in I}$ of differentiable spaces is a differentiable space, and it is the **direct sum** of such family: we have canonical morphisms $j_i\colon X_i \to \coprod_i X_i$ such that, for any differentiable space T, the following map is bijective:

$$\operatorname{Hom}\Big(\coprod_i X_i, T\Big) = \prod_i \operatorname{Hom}(X_i, T) \quad , \quad \varphi \mapsto (\varphi\circ j_i)_{i\in I} \ .$$

Analogously, morphisms of differentiable spaces admit "recollement". If $\{U_i\}_{i\in I}$ is an open cover of a differentiable space X and $\{\varphi_i\colon U_i \to Y\}_{i\in I}$ is a family of morphisms such that $\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}$ for any $i,j\in I$, then there exists a unique morphism $\varphi\colon X\to Y$ such that $\varphi_i = \varphi|_{U_i}$ for any $\in I$.

3.3 Affine Differentiable Spaces

Theorem 3.18. *Let $(X,\mathcal{O}_X)$ be an affine differentiable space. If $(Y,\mathcal{O}_Y)$ is a differentiable space, then the following map is bijective:*

$$\begin{array}{ccc} \operatorname{Hom}(Y,X) & \longrightarrow & \operatorname{Hom}_{\mathbb{R}\text{-alg}}(\mathcal{O}_X(X),\mathcal{O}_Y(Y)) \\ (\varphi,\varphi^*) & \mapsto & \varphi^* \end{array}$$

Proof. Let $X = \operatorname{Spec}_r A$. First we prove that this map is surjective, i.e. that any morphism $j\colon A\to\mathcal{O}_Y(Y)$ comes from a morphism $(\psi,j)\colon Y\to\operatorname{Spec}_r A$.

If $y\in Y$, then the morphism

$$A \xrightarrow{j} \mathcal{O}_Y(Y) \to \mathcal{O}_{Y,y}$$

shows that $j^{-1}(\mathfrak{m}_y)$ is a real ideal of A. So we get a map $\psi\colon Y\to\operatorname{Spec}_r A$, where $x=\psi(y)$ whenever $j(\mathfrak{m}_x)\subseteq\mathfrak{m}_y$. That is to say, if $f\in A$, then $\psi^{-1}(f)_0$ is just the zero-set of the differentiable function $j(f)$, hence it is closed in Y. Therefore, ψ is a continuous map and, for any open set $U\subseteq\operatorname{Spec}_r A$ we have a morphism of $\mathbb{R}$-algebras

$$j:\ A_U \longrightarrow \mathcal{O}_Y(\psi^{-1}U) \quad , \quad j\left(\frac{g}{f}\right) = \frac{j(g)}{j(f)}\ .$$

These morphisms define a morphism of sheaves $j\colon \tilde{A}\to\psi_*\mathcal{O}_Y$ such that the corresponding morphism between the global sections $A\to\mathcal{O}_Y(Y)$ coincides with j. Moreover, the definition of ψ states that the morphisms

$$j_y:\ \tilde{A}_x = A_x \longrightarrow \mathcal{O}_{Y,y}$$

are local, hence $(\psi,j)\colon Y\to\operatorname{Spec}_r A$ is a morphism of differentiable spaces.

Finally, we show that any morphism of differentiable spaces $\varphi\colon Y \to \operatorname{Spec}_r A$ is fully determined by the morphism of $\mathbb{R}$-algebras $\varphi^*\colon A \to \mathcal{O}_Y(Y)$. If $\varphi(y) = x$, then the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi^*} & \mathcal{O}_Y(Y) \\ \downarrow & & \downarrow \\ \widetilde{A}_x = A_x & \xrightarrow{\varphi_y^*} & \mathcal{O}_{Y,y} \end{array}$$

proves, φ_y^* being local, that $\mathfrak{m}_x$ coincides with the inverse image by φ^* of the ideal of all functions in $\mathcal{O}_Y(Y)$ vanishing at y. Hence the continuous map φ is determined by $\varphi^*\colon A \to \mathcal{O}_Y(Y)$. Since A_x is a ring of fractions of A, this commutative square also shows that $\varphi^*\colon A \to \mathcal{O}_Y(Y)$ fully determines the morphisms φ_y^*; hence it determines the morphism of sheaves $\varphi^*\colon\ \widetilde{A} \to \varphi_*\mathcal{O}_Y$.

□

Corollary 3.19. *If X is a differentiable space, then*

$$\operatorname{Hom}(X, \mathbb{R}^n) \ = \ \bigoplus_n \mathcal{O}_X(X)\ .$$

In particular, differentiable functions on X are just morphisms of differentiable spaces $X \to \mathbb{R}$.

Proof. In order to show that the natural map

$$\begin{array}{ccc} \operatorname{Hom}(X, \mathbb{R}^n) & \longrightarrow & \bigoplus_n \mathcal{O}_X(X) \\ (\varphi, \varphi^*) & \mapsto & (\varphi^* x_1, \ldots, \varphi^* x_n) \end{array}$$

is bijective, it is enough to prove it for a basis of open sets in X, since both functors are sheaves on X. If $U = \operatorname{Spec}_r A$ is an affine open set in X, then the map

$$\begin{array}{ccccc} \operatorname{Hom}(U, \mathbb{R}^n) = \operatorname{Hom}_{\mathbb{R}\text{-}alg}(\mathcal{C}^\infty(\mathbb{R}^n), A) & \longrightarrow & \bigoplus_n A \\ (\varphi, \varphi^*) \quad = \quad \varphi^* & \mapsto & (\varphi^* x_1, \ldots, \varphi^* x_n) \end{array}$$

is bijective according to 2.20 and 3.18.

□

Theorem 3.20. *The functors $(X, \mathcal{O}_X) \rightsquigarrow \mathcal{O}_X(X)$ and $A \rightsquigarrow (\operatorname{Spec}_r A, \widetilde{A})$ define an equivalence of the category of affine differentiable spaces with the dual of the category of differentiable algebras.*

Proof. Both functors are mutually inverse by 3.18 and the Localization theorem for differentiable algebras: $\widetilde{A}(X) = A$.

□

3.4 Reduced Differentiable Spaces

Definition. A differentiable space $(X, \mathcal{O}_X)$ is said to be **reduced** if for any open subset $U \subseteq X$ and any differentiable function $f \in \mathcal{O}_X(U)$ we have

$$f = 0 \;\Leftrightarrow\; f(x) = 0, \forall x \in U\ .$$

This is a local concept in the following sense: If a differentiable space X is reduced, then so is every open subset of U. If any point of a differentiable space X has a reduced open neighbourhood, then X is reduced.

By definition, if a differentiable space X is reduced, then the natural morphism $\mathcal{O}_X(U) \to \mathcal{C}(U, \mathbb{R})$, $f \mapsto \hat{f}$, is injective for any open set $U \subseteq X$. Therefore the structural sheaf $\mathcal{O}_X$ is canonically isomorphic to a subsheaf of the sheaf $\mathcal{C}_X$ of real-valued continuous functions on X. That is to say, any differentiable function $f \in \mathcal{O}_X(U)$ may be understood (as we shall always do) as a continuous map $f\colon U \to \mathbb{R}$, so that $(X, \mathcal{O}_X)$ is a reduced ringed space.

Let $(\varphi, \varphi^*)\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a morphism between reduced differentiable spaces. The morphism of sheaves $\varphi^*\colon \mathcal{O}_X \to \varphi_*\mathcal{O}_Y$ is fully determined by the continuous map $\varphi\colon Y \to X$. In fact, if $x = \varphi(y)$, then the condition of $\varphi^*_y\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ being a local morphism means that we have $(\varphi^* f_x)(y) = f_x(\varphi(y))$ for any germ $f_x \in \mathcal{O}_{X,x}$. Hence, if $f \in \mathcal{O}_X(U)$, then the differentiable function $\varphi^* f \in \mathcal{O}_Y(\varphi^{-1}U)$ is just the composition of f and φ:

$$\varphi^*(f) \;=\; f \circ \varphi\ ,$$

so that φ is in fact a morphism of reduced ringed spaces. Therefore, *in the reduced case, morphisms of differentiable spaces are just morphisms of reduced ringed spaces.*

Example 3.21. Let Y be a closed set in a differentiable space $(X, \mathcal{O}_X)$ and let $\mathcal{I}_Y$ be the sheaf of differentiable functions vanishing at Y. Then the locally ringed space $(Y, \mathcal{O}_X/\mathcal{I}_Y)$ is a reduced differentiable space. In fact, the problem being local, we may assume $X = \operatorname{Spec}_r A$. Let $\mathfrak{p}_Y$ be the ideal in A of all differentiable functions vanishing at Y. We know that $A/\mathfrak{p}_Y$ is a differentiable algebra and that $\operatorname{Spec}_r A/\mathfrak{p}_Y = Y$ (see 2.31). Since $\mathfrak{p}_Y = \mathcal{I}_Y(X)$ we have $\widetilde{\mathfrak{p}}_Y = \mathcal{I}_Y$ (by 3.11) and then $\mathcal{O}_X/\mathcal{I}_Y = \widetilde{A}/\widetilde{\mathfrak{p}}_Y = (A/\mathfrak{p}_Y)^{\sim}$. Therefore Y is the reduced differentiable space defined by the differentiable algebra $A/\mathfrak{p}_Y$. In conclusion, *any closed subset of a differentiable space inherits a natural structure of reduced differentiable space.*

When $Y = X$, we obtain a reduced differentiable space and we put

$$X_{red} \;:=\; (X, \mathcal{O}_X/\mathcal{I}_X)\ .$$

If U is an affine open set in X, then we have

$$\mathcal{O}_{X_{red}}(U) = \;\mathcal{O}_X(U)/\mathcal{I}_X(U) \;=\; \mathcal{O}_X(U)_{red}\ ,$$
$$\mathcal{I}_X(U) = \;\{f \in \mathcal{O}_X(U)\colon\ \hat{f} = 0\}\ .$$

If $(\varphi, \varphi^*)\colon Y \to X$ is a morphism of differentiable spaces, then we have $\varphi^*(\mathcal{I}_X) \subseteq \varphi_*\mathcal{I}_Y$, so that $\varphi^*\colon \mathcal{O}_X \to \varphi_*\mathcal{O}_Y$ induces a morphism of sheaves of rings $\varphi^*_{red}\colon \mathcal{O}_X/\mathcal{I}_X \to \varphi_*(\mathcal{O}_Y/\mathcal{I}_Y)$ and we obtain a morphism of differentiable spaces

$$\varphi_{red} := (\varphi, \varphi^*_{red})\colon Y_{red} \longrightarrow X_{red} .$$

Moreover, the canonical projection $\pi\colon \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_X$ defines a morphism $i := (Id, \pi)\colon X_{red} \to X$, which is an isomorphism whenever X is reduced. The following square is commutative:

$$\begin{array}{ccc} Y_{red} & \xrightarrow{\varphi_{red}} & X_{red} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{\varphi} & X \end{array}$$

so that, if Y is reduced, then any morphism of differentiable spaces $Y \to X$ factors through the canonical morphism $i\colon X_{red} \to X$.

Definition. Let Z be a topological subspace of $\mathbb{R}^n$. A continuous function $f\colon Z \to \mathbb{R}$ is said to be of class $\mathcal{C}^\infty$ if any point $z \in Z$ has an open neighbourhood U_z in $\mathbb{R}^n$ such that f coincides on $Z \cap U_z$ with the restriction of some $\mathcal{C}^\infty$-function on U_z. Functions of class $\mathcal{C}^\infty$ on Z form a ring denoted by $\mathcal{C}^\infty(Z)$. So we obtain a sheaf $\mathcal{C}^\infty_Z$ of continuous functions on Z:

$$\mathcal{C}^\infty_Z(V) := \mathcal{C}^\infty(V) \quad , \qquad V \text{ open set in } Z .$$

Proposition 3.22. *Let $\mathfrak{a}$ be a closed ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$. If the affine differentiable space $Y = \operatorname{Spec}_r(\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a})$ is reduced, then $\mathcal{O}_Y = \mathcal{C}^\infty_Y$. That is to say, for any open set $V \subseteq Y$ we have*

$$\mathcal{O}_Y(V) \;=\; \mathcal{C}^\infty(V) .$$

Proof. By the Localization theorem for differentiable algebras, any differentiable function $f \in \mathcal{O}_Y(V)$ is a quotient $f = g/h$, where $g, h \in \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ and h does not vanish at any point of V. These functions are restrictions of some differentiable functions on $\mathbb{R}^n$; hence $f \in \mathcal{C}^\infty(V)$ and we obtain that $\mathcal{O}_Y$ is a subsheaf of $\mathcal{C}^\infty_Y$. In order to prove the coincidence, we have to show that $\mathcal{O}_{Y,y} = \mathcal{C}^\infty_{Y,y}$ at any point $y \in Y$. Now, the inclusion map $\mathcal{O}_{Y,y} \hookrightarrow \mathcal{C}^\infty_{Y,y}$ is surjective because so is (see 1.6) the natural morphism $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty_{Y,y}$. □

Resuming, we have proved the following

Theorem 3.23. *The category of reduced differentiable spaces is equivalent to the category of reduced ringed spaces $(X, \mathcal{O}_X)$ satisfying the following condition:*

Each point $x \in X$ has an open neighbourhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(Y, \mathcal{C}^\infty_Y)$ for some closed subset Y of an affine space $\mathbb{R}^n$.

4 Topology of Differentiable Spaces

The definition of differentiable space is a purely local concept, so that differentiable spaces may not be paracompact or even separated. However, they have the same local properties as closed sets in affine spaces, because any affine open set is homeomorphic to a closed set in some $\mathbb{R}^n$. Therefore differentiable spaces are locally compact, locally metrizable and so on. In this chapter we recall some well-known results on the covering dimension. We use them, in the affine case, to prove that the category of locally free sheaves of bounded rank is equivalent to the category of finitely generated projective modules over the ring of global differentiable functions. We also use them in the proof of the embedding theorem in chapter 5.

4.1 Partitions of Unity

Proposition 4.1. *Let X be a differentiable space.*

1. *Any point of X is closed.*
2. *X is locally compact.*
3. *If X is separated, then it is regular (open sets separate points and closed sets).*
4. *If X is separated and its topology has a countable basis, then X is metrizable.*

Proof. (1) Any point is locally closed, since so is any point of an affine open set, hence it is closed.

(2) Every closed subset of $\mathbb{R}^n$ is locally compact.

(3) Let Y be a closed set in X and let $x \in X - Y$. If K is a compact neighbourhood of x such that $K \cap Y = \emptyset$, then K is closed in X because X is assumed to be separated. Therefore $\overset{\circ}{K}$ and $X - K$ separate x and Y.

(4) According to the metrization theorem of Urysohn (see [34]) any regular space with a countable basis of open sets is metrizable.

□

Definition. Let f be a differentiable function on a differentiable space X. The **support** of f is defined to be its support as a section of the sheaf $\mathcal{O}_X$:

$$\operatorname{Supp} f \;:=\; \{x \in X :\; f_x \neq 0\}\;.$$

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Definition. Let $\{U_i\}_{i\in I}$ be an open cover of a differentiable space X. We say that a family $\{\phi_i\}_{i\in I}$ of differentiable functions on X is a **partition of unity** subordinated to $\{U_i\}_{i\in I}$ if it satisfies the following conditions:

1. $\operatorname{Supp}\phi_i \subseteq U_i$ and $\phi_i \geq 0$ (i.e. $\phi_i(x) \geq 0$, $\forall x \in X$) for any index $i \in I$.
2. The family $\{\operatorname{Supp}\phi_i\}_{i\in I}$ is locally finite.
3. $\sum_i \phi_i = 1$ (i.e., both terms have the same germ at any point $x \in X$, the sum being finite by 2).

The standard arguments in the smooth case prove the existence of partitions of unity:

Lemma 4.2. *Let K be a compact set in a separated differentiable space X and let U be a neighbourhood of K in X. There exists a global differentiable function $h \in \mathcal{O}_X(X)$ such that*

1. *$h = 1$ on an open neighbourhood of K.*
2. *$\operatorname{Supp} h \subseteq U$.*
3. *$0 \leq h(x) \leq 1$ at any point $x \in X$.*

Corollary 4.3. *Let U be an open set in a separated differentiable space X and let $x \in U$. If f is a differentiable function on U, then the germ of f at x coincides with the germ of some global differentiable function.*

Theorem of existence of partitions of unity. *Let X be a separated differentiable space whose topology has a countable basis. Any open cover of X has a subordinated partition of unity.*

Corollary 4.4. *Let $\{U_i\}_{i\in I}$ be an open cover of a separated differentiable space X whose topology has a countable basis. There exists a locally finite open cover $\{V_i\}_{i\in I}$ of X such that $\overline{V}_i \subseteq U_i$.*

Proof. Let $\{\phi_i\}_{i\in I}$ be a partition of unity subordinated to $\{U_i\}_{i\in I}$. Then

$$V_i := \{x \in X\colon\ \phi_i(x) \neq 0\}$$

satisfies the required conditions, since $\overline{V}_i \subseteq \operatorname{Supp}\phi_i \subseteq U_i$.

□

Corollary 4.5. *Let X be a separated differentiable space whose topology has a countable basis. There exists a differentiable function $\phi\colon X \to \mathbb{R}$ which is proper as a continuous map (the fibre of any compact set is compact) and, in particular, it is a closed map.*

Proof. Let $\{U_n\}$ be a locally finite countable cover of X by open sets with compact closure. By 4.4 there exists an open cover $\{V_n\}$ such that $\overline{V}_n \subseteq U_n$ for any index n. By 4.2 there exists a global differentiable function $\phi_n \geq 0$ such that $\operatorname{Supp}\phi_n \subseteq U_n$ and $\phi_n = 1$ on a neighbourhood of $\overline{V}_n$. Let us consider the differentiable function $\phi = \sum_n n\phi_n$. If $c \in \mathbb{R}$, then the closed set $\phi \leq c$ is compact, since it is contained in the compact set $\overline{V}_1 \cup \ldots \cup \overline{V}_n$, where $n \geq c$.

□

Corollary 4.6. *Let X be a separated differentiable space whose topology has a countable basis. If Y, Z are disjoint closed subsets, then there exists a global differentiable function $0 \leq \phi \leq 1$ such that $\phi = 1$ on an open neighbourhood of Z and $\phi = 0$ on an open neighbourhood of Y.*

Proof. Let $\{\phi := \phi_1, \phi_2\}$ be a partition of unity subordinated to the open cover $\{U_1 = X - Y, U_2 = X - Z\}$.

□

Corollary 4.7. *Let X be a separated differentiable space whose topology has a countable basis. If $\mathcal{M}$ is an $\mathcal{O}_X$-module, then $H^p(X, \mathcal{M}) = 0$ for any $p > 0$.*

Proof. X is paracompact (4.4) and the sheaf of rings $\mathcal{O}_X$ is soft because it admits partitions of unity. We conclude by theorems II 3.7.1, 3.7.2 and 4.4.3 of [15].

□

4.2 Covering Dimension

Definition. The **order** of a cover is defined to be the greatest natural number d, if it exists, such that there are $d+1$ members of the cover with non-empty intersection; otherwise the order is defined to be infinite.

Let us recall that a cover $\mathcal{R}'$ is a **refinement** of another cover $\mathcal{R}$ of the same space if for every $U' \in \mathcal{R}'$ there exists $U \in \mathcal{R}$ such that $U' \subseteq U$.

The (covering) **dimension** of a normal topological space X is defined to be the least natural number d, if it exists, such that any finite open cover of X has a finite open refinement of order $\leq d$; otherwise the dimension of X is defined to be infinite. We denote it by $\dim X$.

It is clear that $\dim Y \leq \dim X$ for any closed set Y in X. The fundamental theorem of dimension theory states that $\dim \mathbb{R}^d = d$ ([20] Theorems IV.1 and V.8).

Let $\mathcal{R} = \{U_i\}$ be an open cover of a topological space X. Given a subset $Y \subseteq X$, the restriction of $\mathcal{R}$ to Y is defined to be $\mathcal{R}_{|Y} := \{U_i \cap Y\}$.

Lemma 4.8. *Let X be a separated space and let $K \subseteq X$ be a compact subset of dimension $\leq d$. Any open cover $\mathcal{R} = \{U_i\}$ of X has an open refinement $\mathcal{R}' = \{V_i\}$ such that*

1. *$V_i \subseteq U_i$ for any index i.*
2. *$\mathcal{R}'_{|K}$ has order $\leq d$ and $\mathcal{R}'_{|X-K} = \mathcal{R}_{|X-K}$.*

Proof. Since K is assumed to be compact, there exists a finite open refinement $\{W'_j\}$ of $\mathcal{R}_{|K} = \{U_i \cap K\}$. Since $\dim K \leq d$, we may assume that $\{W'_j\}$ has order $\leq d$. For each j let us choose an index $\sigma(j)$ such that $W'_j \subseteq U_{\sigma(j)} \cap K$. Let W_j be an open subset of $U_{\sigma(j)}$ such that $W'_j = W_j \cap K$. Then the desired refinement is $\mathcal{R}' = \{V_i\}$ where

$$V_i := (U_i - U_i \cap K) \cup \bigcup_{\sigma(j)=i} W_j \ .$$

□

Remark 4.9. The same argument proves the following variation: *Let X be a normal space and let $Y \subseteq X$ be a closed subset of dimension $\leq d$. Any finite open cover $\mathcal{R} = \{U_i\}$ of X has a finite open refinement $\mathcal{R}' = \{V_i\}$ such that*

1. *$V_i \subseteq U_i$ for any index i.*
2. *$\mathcal{R}'_{|Y}$ has order $\leq d$ and $\mathcal{R}'_{|X-Y} = \mathcal{R}_{|X-Y}$.*

Corollary 4.10. *Let X be a normal space such that $X = Y_1 \cup Y_2$, where Y_1, Y_2 are closed subspaces of dimension $\leq d$. Then $\dim X \leq d$.*

Proof. Let $\mathcal{R} = \{U_i\}$ be a finite open cover of X. By 4.9 there exists a finite open refinement $\mathcal{R}' = \{U'_i\}$ of $\mathcal{R}$ such that $\mathcal{R}'_{|Y_1}$ has order $\leq d$. Again by 4.9, there exists a finite open refinement $\mathcal{R}'' = \{U''_i\}$ of $\mathcal{R}'$ such that $\mathcal{R}''_{|Y_2}$ has order $\leq d$. It is immediate that $\mathcal{R}''$ has order $\leq d$.

□

Lemma 4.11. *Let X be a separated space whose topology has a countable basis. Let us assume that any point of X has a compact neighbourhood of dimension $\leq d$. Then there exists a countable family $\{K_n\}$ of compact subsets of dimension $\leq d$ such that:*

$$X = \bigcup_n K_n \qquad , \qquad K_n \subseteq \overset{\circ}{K}_{n+1} \ .$$

Proof. We may consider a countable open basis $\{V_n\}$ such that $\overline{V}_n$ is a compact subset of dimension $\leq d$ for any index n. The family $\{K_n\}$ is defined recursively: We write $K_1 := \overline{V}_1$. Let us assume that K_{n-1} has been constructed and let us choose a finite family of indexes $n_1, \ldots, n_r$ such that $K_{n-1} \subseteq V_{n_1} \cup \cdots \cup V_{n_r}$; then we define $K_n := \overline{V}_n \cup \overline{V}_{n_1} \cup \cdots \cup \overline{V}_{n_r}$.

□

Theorem 4.12. *Let X be a separated space whose topology has a countable basis. Let us assume that any point of X has a compact neighbourhood of dimension $\leq d$. Then any open cover $\mathcal{R} = \{U_i\}$ of X has an open refinement $\mathcal{R}' = \{U'_i\}$ of order $\leq d$. Therefore, $\dim X \leq d$.*

Proof. With the notations of 4.11, let $Q_n := K_n - \overset{\circ}{K}_{n-1}$, which are compact subsets of dimension $\leq d$ and $X = \bigcup_n Q_n$. By 4.8 there exists an open refinement $\mathcal{R}^1 = \{U_i^1\}$ of $\mathcal{R} = \{U_i\}$ such that $\mathcal{R}^1_{|Q_1}$ has order $\leq d$ and $\mathcal{R}^1_{|X-Q_1} = \mathcal{R}_{|X-Q_1}$. We define recursively an open refinement $\mathcal{R}^n = \{U_i^n\}$ of $\mathcal{R}^{n-1}$ such that $\mathcal{R}^n_{|Q_n}$ has order $\leq d$ and $\mathcal{R}^n_{|X-Q_n} = \mathcal{R}^{n-1}_{|X-Q_n}$. Let $\mathcal{R}^\infty = \{U_i^\infty\}$ where $U_i^\infty = \bigcap_n U_i^n$. It easy to check that $\mathcal{R}^\infty$ is an open refinement of $\mathcal{R}$ of order $\leq d$.

□

Theorem 4.13. *Let X be a separated differentiable space whose topology has a countable basis. If any point of X has a neighbourhood of dimension $\leq d$, then any open cover $\{U_i\}$ of X has a locally finite open refinement which is a union of $d+1$ families of disjoint open sets.*

Proof. By 4.12 we may assume that $\{U_i\}$ has order $\leq d$. Let $\{\phi_i\}$ be a partition of unity subordinated to $\{U_i\}$, so that the family $\{\operatorname{Supp}\phi_i\}$ has order $\leq d$.

For any finite set of indices $\{i_0,\ldots,i_n\}$ we consider the open set

$$U(i_0,\ldots,i_n) := \{x \in X\colon\ \phi_i(x) < \min(\phi_{i_0}(x),\ldots,\phi_{i_n}(x)),\ \forall i \neq i_0,\ldots,i_n\}\ .$$

$F = \{U(i_0,\ldots,i_n)\}$ is an open cover of X, because $\sum_i \phi_i = 1$, it refines $\{U_i\}$, because $U(i_0,\ldots,i_n) \subseteq \operatorname{Supp}\phi_{i_0} \subseteq U_{i_0}$, and it is a locally finite family because any point of X has an open neighbourhood U where all the functions ϕ_i, up to a finite number $\phi_{j_1},\ldots,\phi_{j_r}$, vanishes. Hence $U(i_0,\ldots,i_n)$ may intersect U only when $\{i_0,\ldots,i_n\} \subseteq \{j_1,\ldots,j_r\}$.

Given $0 \leq n \leq d$, let F_n be the family of all open sets $U(i_0,\ldots,i_n)$, which are disjoint. In fact, if $\{i_0,\ldots,i_n\} \neq \{j_0,\ldots,j_n\}$, let i be an index of the first set which is not in the second set, and let j be an index of the second set which is not in the first set. Then $\phi_j < \phi_i$ on $U(i_0,\ldots,i_n)$, and $\phi_i < \phi_j$ on $U(j_0,\ldots,j_n)$; hence $U(i_0,\ldots,i_n) \cap U(j_0,\ldots,j_n) = \emptyset$.

Finally, note that $F = F_0 \cup F_1 \cup \ldots \cup F_d$ since, given a point $x \in X$, there are no more than $d+1$ indices i such that $\phi_i(x) > 0$.

□

Definition. Let $(X,\mathcal{O})$ be a ringed space. An $\mathcal{O}$-module $\mathcal{E}$ is said to be **locally free of finite rank** if any point $x \in X$ has an open neighbourhood U such that $\mathcal{E}|_U \simeq (\mathcal{O}|_U)^r$ for some natural number r. In such a case we say that $\mathcal{E}$ is **trivial** over U of **rank** r.

Lemma 4.14. *Let A be a differentiable algebra and let $X = \operatorname{Spec}_r A$. The functors $\mathcal{E} \rightsquigarrow \mathcal{E}(X)$ and $P \rightsquigarrow \widetilde{P}$ define an equivalence of the category of locally free $\widetilde{A}$-modules of finite rank, trivial over some finite open cover of X, with the category of finitely generated projective A-modules.*

Proof. If P is a finitely generated projective A-module, then ([5] II.2.1) there exists a finite open cover $\{V_1,\ldots,V_r\}$ of $\operatorname{Spec} A$ such that $P_{V_i} \simeq (A_{V_i})^{n_i}$. Since $X = \operatorname{Spec}_r A \subseteq \operatorname{Spec} A$, we have $P_{U_i} \simeq (A_{U_i})^{n_i}$, where $U_i := V_i \cap X$, and we conclude that

$$\widetilde{P}\,|_{U_i} \;=\; (P_{U_i})^{\sim} \simeq (A_{U_i}^{n_i})^{\sim} \;=\; \left(\widetilde{A}\,|_{U_i}\right)^{n_i}\ .$$

That is to say, $\widetilde{P}$ is trivial of finite rank over the finite open cover $\{U_1,\ldots,U_r\}$ of X. Moreover, P being an A-module of finite presentation, 3.12 shows that we have $P = \widetilde{P}\,(X)$.

Conversely, if an $\widetilde{A}$-module $\mathcal{E}$ is trivial over a finite open cover $\{U_1, \ldots, U_r\}$ of X, then we choose an open cover $\{V_1, \ldots, V_r\}$ such that $\overline{V_i} \subseteq U_i$, and global sections $\psi_i \in \widetilde{A}\,(X)$ such that $\psi_i = 1$ on V_i and $\operatorname{Supp}\psi_i \subseteq U_i$. Since we have $\mathcal{E}|_{U_i} \simeq (\widetilde{A}\,|_{U_i})^{n_i}$, there exist sections $s_{i1}, \ldots, s_{in_i} \in \mathcal{E}(U_i)$ which generate the stalk $\mathcal{E}_x$ at any point $x \in U_i$; hence $\psi_i s_{i1}, \ldots, \psi_i s_{in_i}$ are global sections of $\mathcal{E}$ which generate the stalk $\mathcal{E}_x$ at any point $x \in V_i$. We conclude that a finite number of global sections generate the stalk at any point of X. That is to say, we have a surjective morphism of sheaves $p\colon (\widetilde{A})^n \to \mathcal{E}$.

By 3.11, the morphism $\widetilde{A}\,(U_i)^n \to \mathcal{E}(U_i) \simeq \widetilde{A}\,(U_i)^{n_i}$ is surjective for any index i, so that it has some section; hence the morphism of sheaves $(\widetilde{A})^n|_{U_i} \to \mathcal{E}|_{U_i}$ has some section $\sigma_i\colon \mathcal{E}|_{U_i} \to (\widetilde{A})^n|_{U_i}$. It follows that p has a section $\sum_i \psi_i \sigma_i$. Therefore $(\widetilde{A})^n \simeq \mathcal{E} \oplus \mathcal{K}$ and we conclude that $A^n = (\widetilde{A})^n(X) \simeq \mathcal{E}(X) \oplus \mathcal{K}(X)$, so that $\mathcal{E}(X)$ is a finitely generated projective A-module. Moreover, we have $\mathcal{E} = \mathcal{E}(X)^{\sim}$ by 3.11.

□

Corollary 4.15. *Let A be a differentiable algebra with compact real spectrum. The category of finitely generated projective A-modules is equivalent to the category of locally free $\widetilde{A}$-modules of finite rank.*

Theorem 4.16. *Let A be a differentiable algebra and let $X = \operatorname{Spec}_r A$. The functors $\mathcal{E} \rightsquigarrow \mathcal{E}(X)$ and $P \rightsquigarrow \widetilde{P}$ define an equivalence of the category of locally free $\widetilde{A}$-modules of bounded rank with the category of finitely generated projective A-modules.*

Proof. According to 4.14, it is enough to show that any locally free $\widetilde{A}$-module of bounded rank is trivial over a finite open cover of X. Now, any $\widetilde{A}$-module which is trivial of constant rank over a family of disjoint open sets is also trivial over the union, and we conclude by 4.13.

□

5 Embeddings

In this chapter we introduce the natural notion of differentiable subspace of a differentiable space. Affine differentiable spaces are just differentiable spaces which are isomorphic to some closed differentiable subspace of an affine space $\mathbb{R}^n$. The main result is the embedding theorem which provides a characterization of affine differentiable spaces: They are separated differentiable spaces of bounded embedding dimension with a countable basis of open sets. In particular, compact separated differentiable spaces are always affine.

5.1 Differentiable Subspaces

Definition. Let $(X, \mathcal{O}_X)$ be a differentiable space. Given a locally closed subspace Y of X and a sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X|_Y$, let us put $\mathcal{O}_X/\mathcal{I} := (\mathcal{O}_X|_Y)/\mathcal{I}$. When $(Y, \mathcal{O}_X/\mathcal{I})$ is a differentiable space, we say that it is a **differentiable subspace** of $(X, \mathcal{O}_X)$. It is said to be an **open differentiable subspace** if Y is open in X and $\mathcal{I} = 0$. It is said to be **closed** if Y is closed in X.

By definition the topological subspace Y underlying a differentiable subspace of X is locally closed in X, but we shall prove in 5.27 that this condition is redundant.

Each differentiable subspace $(Y, \mathcal{O}_Y)$ of a differentiable space $(X, \mathcal{O}_X)$ is endowed with a canonical morphism $(i, i^*)\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ called the **inclusion morphism**, where $i\colon Y \hookrightarrow X$ is the inclusion map and the morphism of sheaves $i^*\colon \mathcal{O}_X|_Y \to \mathcal{O}_Y := (\mathcal{O}_X|_Y)/\mathcal{I}$ is defined to be the canonical projection. If $f \in \mathcal{O}_X(X)$, then we say that $i^*f \in \mathcal{O}_Y(Y)$ is the **restriction** of the differentiable function f to the differentiable subspace Y, and we put $f|_Y := i^*f$. If $\psi\colon X \to Z$ is a morphism of differentiable spaces, then $\psi|_Y := \psi \circ i$ is said to be the **restriction** of the morphism ψ to the differentiable subspace Y.

Note that the restriction morphism $i^*\colon \mathcal{O}_X|_Y \to \mathcal{O}_Y$ is surjective; i.e., the restriction morphism $i^*\colon \mathcal{O}_{X,y} \to \mathcal{O}_{Y,y}$ is surjective at any point $y \in Y$.

Definition. A differentiable subspace $(Y_1, \mathcal{O}_X/\mathcal{I}_1)$ of a differentiable space $(X, \mathcal{O}_X)$ is said to be **contained** in another differentiable subspace $(Y_2, \mathcal{O}_X/\mathcal{I}_2)$ when $Y_1 \subseteq Y_2$ and $\mathcal{I}_2|_{Y_1} \subseteq \mathcal{I}_1$. The inclusion relation between differentiable subspaces is denoted with the symbol $\subseteq$ and it is an order relation in the set of all differentiable subspaces of $(X, \mathcal{O}_X)$.

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It is easy to check that $(Y_1, \mathcal{O}_X/\mathcal{I}_1) \subseteq (Y_2, \mathcal{O}_X/\mathcal{I}_2)$ if and only if the inclusion morphism $i_1 \colon Y_1 \hookrightarrow X$ factors through the inclusion morphism $i_2 \colon Y_2 \hookrightarrow X$; that is to say, if it exists a morphism $j \colon Y_1 \to Y_2$ such that $i_1 = i_2 j$. Such morphism j is unique.

Any differentiable subspace of a differentiable subspace of $(X, \mathcal{O}_X)$ may be canonically identified with a differentiable subspace of $(X, \mathcal{O}_X)$.

Remark 5.1. Let $(X, \mathcal{O}_X)$ be a differentiable space. According to 3.14, any open subset $U \subseteq X$ inherits a natural structure of differentiable space, defined by the sheaf $\mathcal{O}_X|_U$. It is clear that $(U, \mathcal{O}_X|_U)$ is an open differentiable subspace of $(X, \mathcal{O}_X)$, and so is obtained any open differentiable subspace of $(X, \mathcal{O}_X)$.

Let $(Y, \mathcal{O}_X/\mathcal{I})$ be a differentiable subspace of X. Even if Y is open in X, it may be that $(Y, \mathcal{O}_X/\mathcal{I})$ is not an open differentiable subspace of X (since it may be that $\mathcal{I} \neq 0$) as show the differentiable subspaces of $\operatorname{Spec}_r \mathbb{R}[[\,t\,]]$ defined by the ideals (t^{r+1}). However, this phenomenon does not occur when X is reduced, because $\mathcal{I}$ is always a sheaf of differentiable functions vanishing at any point of Y; hence $\mathcal{I} = 0$ whenever Y is open and X is reduced.

Remark 5.2. Let $(X, \mathcal{O}_X)$ be a differentiable space and let $\mathcal{J}$ be a sheaf of ideals of $\mathcal{O}_X$. Let us consider the support $Y = \{x \in X \colon \ \mathcal{J}_x \neq \mathcal{O}_{X,x}\}$ of the quotient sheaf $\mathcal{O}_X/\mathcal{J}$, which is closed in X, so that $\mathcal{O}_X/\mathcal{J}$ may be considered as a sheaf on Y, namely $(\mathcal{O}_X/\mathcal{J})|_Y$. If $(Y, \mathcal{O}_X/\mathcal{J})$ is a differentiable space, then it is a closed differentiable subspace of $(X, \mathcal{O}_X)$. We say that it is the closed differentiable subspace of X defined by the sheaf of ideals $\mathcal{J}$ and we denote it by

$$\operatorname{Spec}_r (\mathcal{O}_X/\mathcal{J}) \ := \ (Y, \mathcal{O}_X/\mathcal{J}) \ .$$

Any closed differentiable subspace of X is defined by a unique sheaf of ideals of $\mathcal{O}_X$, because sheaves of ideals $\mathcal{I}$ of $i^*\mathcal{O}_X = \mathcal{O}_X|_Y$ correspond to sheaf of ideals $\mathcal{J}$ of $\mathcal{O}_X$ which coincide with $\mathcal{O}_X$ on the open set $X - Y$.

If $\mathcal{J}_1$ and $\mathcal{J}_2$ are sheaf of ideals of $\mathcal{O}_X$, corresponding to certain closed differentiable subspaces Y_1, Y_2 of X, then we have $Y_1 \subseteq Y_2 \Leftrightarrow \mathcal{J}_1 \supseteq \mathcal{J}_2$.

Remark 5.3. Any differentiable subspace is a closed differentiable subspace of an open subspace. Let $(Y, \mathcal{O}_Y)$ be a differentiable subspace of a differentiable space $(X, \mathcal{O}_X)$. Since Y is locally closed in X we have that Y is a closed subset of some open set $U \subseteq X$. Let $\mathcal{O}_Y = (\mathcal{O}_X|_Y)/\mathcal{I}$. Since $\mathcal{O}_U = \mathcal{O}_X|_U$ and $\mathcal{O}_X|_Y = \mathcal{O}_U|_Y$, it follows that $\mathcal{I}$ is a sheaf of ideals of $\mathcal{O}_U|_Y$ such that $(Y, \mathcal{O}_U/\mathcal{I})$ is a differentiable space; that is to say, $(Y, \mathcal{O}_Y)$ is a closed differentiable subspace of $(U, \mathcal{O}_U)$.

Remark 5.4. Let $\mathcal{V}$ be a smooth manifold. If $\mathcal{W}$ is a closed smooth submanifold of $\mathcal{V}$, let us consider the sheaf of ideals $\mathcal{I}_\mathcal{W}$ of $\mathcal{C}^\infty_\mathcal{V}$ defined by the differentiable functions vanishing on $\mathcal{W}$. It is clear that $\mathcal{C}^\infty_\mathcal{W} = \mathcal{C}^\infty_\mathcal{V}/\mathcal{I}_\mathcal{W}$, hence $\mathcal{I}_\mathcal{W}$ defines a closed differentiable subspace of $(\mathcal{V}, \mathcal{C}^\infty_\mathcal{V})$, which is canonically isomorphic to $(\mathcal{W}, \mathcal{C}^\infty_\mathcal{W})$.

In general, if $\mathcal{W}$ is a smooth submanifold of $\mathcal{V}$, then $\mathcal{W}$ is a closed smooth submanifold of an open set $U \subseteq \mathcal{V}$, so that $\mathcal{W}$ defines a closed differentiable subspace of $(U, \mathcal{C}^\infty_U)$, hence a differentiable subspace of $(\mathcal{V}, \mathcal{C}^\infty_\mathcal{V})$, which is canonically isomorphic to $(\mathcal{W}, \mathcal{C}^\infty_\mathcal{W})$.

Remark 5.5. Let $(X, \mathcal{O}_X)$ be a differentiable space and let Y be a closed set in X. Let us consider the sheaf $\mathcal{I}_Y$ of all differentiable functions vanishing at any point of Y. We know that $(Y, \mathcal{O}_X/\mathcal{I}_Y)$ is a reduced differentiable space (see 3.21). Therefore $\mathcal{I}_Y$ defines a reduced closed differentiable subspace of X. Any reduced closed differentiable subspace of X is defined by the sheaf of ideals $\mathcal{I}_Y$ of a unique closed set $Y \subseteq X$.

When $Y = X$, such differentiable subspace is just X_{red} (see 3.21).

5.2 Universal Properties

Proposition 5.6. *Let I be a closed ideal of a differentiable algebra A. Then* $\operatorname{Spec}_r(A/I)$ *is a closed differentiable subspace of* $\operatorname{Spec}_r A$. *Conversely, any closed differentiable subspace of* $\operatorname{Spec}_r A$ *is defined by a unique closed ideal of A.*

Proof. Since I is a closed ideal we have that A/I is a differentiable algebra. Let $i\colon \operatorname{Spec}_r(A/I) = (I)_0 \hookrightarrow \operatorname{Spec}_r A$ be the morphism induced by the canonical projection $A \to A/I$. The corresponding morphism of sheaves $\widetilde{A} \to (A/I)^{\sim}$ is surjective and its kernel is the sheaf of ideals $\widetilde{I}$. The support of $\widetilde{A}/\widetilde{I} = (A/I)^{\sim}$ is the closed set $(I)_0$, so that $((I)_0, \widetilde{A}/\widetilde{I}) \simeq \operatorname{Spec}_r(A/I)$ is a closed differentiable subspace of $\operatorname{Spec}_r A$.

Conversely, given a closed differentiable subspace $(Y, \widetilde{A}/\mathcal{I})$ of $\operatorname{Spec}_r A$, let I be the ideal of all global sections of $\mathcal{I}$. By definition there exists an affine open cover $\{U_i\}$ of Y, hence $U_i = \operatorname{Spec}_r \mathcal{O}_Y(U_i)$. The kernel I_i of the natural morphism $A \to \mathcal{O}_Y(U_i)$ is a closed ideal by 2.24. Therefore $I = \bigcap I_i$ is a closed ideal of A and 3.11 states that $\mathcal{I} = \widetilde{I}$, so that $(Y, \widetilde{A}/\mathcal{I} = (A/I)^{\sim}) = \operatorname{Spec}_r(A/I)$ is the closed differentiable subspace defined by I.

The uniqueness is obvious, since the ideal I must be the kernel of the morphism of rings $i^*\colon A \to \mathcal{O}_Y(Y)$.

□

Corollary 5.7. *Any affine differentiable space is isomorphic to some closed differentiable subspace of an affine space $\mathbb{R}^n$.*

Corollary 5.8. *Any differentiable subspace of an affine differentiable space is affine.*

Proof. By 3.14, any open differentiable subspace of an affine differentiable space is affine. By 5.6, any closed differentiable subspace of an affine differentiable space is affine. We conclude, since any differentiable subspace is a closed differentiable subspace of some open differentiable subspace (see 5.3).

□

Corollary 5.9. *Let $(X, \mathcal{O}_X)$ be a differentiable space and let $\{U_i\}$ be an affine open cover of X. A sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$ defines a closed differentiable subspace of X if and only if each ideal $\mathcal{I}(U_i)$ is closed in the differentiable algebra $\mathcal{O}_X(U_i)$.*

Proof. The direct implication follows from 5.6.

Conversely, we have $\mathcal{I}|_{U_i} = \mathcal{I}(U_i)^{\sim}$ by 3.11. If $\mathcal{I}(U_i)$ is a closed ideal, then $\mathcal{I}|_{U_i}$ defines a closed differentiable subspace of U_i and we conclude that $\mathcal{I}$ defines a closed differentiable subspace of X.

□

Corollary 5.10. *Let Y be a closed subset of a differentiable space X. The* ***sheaf of Whitney ideals***

$$\mathcal{W}_{Y/X} := \bigcap_{y,r} m_y^{r+1} \qquad (y \in Y,\ r \in \mathbb{N})$$

defines a closed differentiable subspace $\mathbf{W}_{Y/X}$ of X, called the ***Whitney subspace*** *of Y in X, which contains any other closed differentiable subspace $(Z, \mathcal{O}_X/\mathcal{I})$ such that Z is a subset of Y.*

Proof. If U is an affine open subspace of X, then $\mathcal{W}_{Y/X}(U)$ coincides with the Whitney ideal $W_{(Y\cap U)/U}$, which is a closed ideal of the differentiable algebra $\mathcal{O}_X(U)$. Hence $\mathcal{W}_{Y/X}$ defines a closed differentiable subspace of X by 5.9.

Finally, the Spectral theorem 2.28 shows that $W_{(Y\cap U)/U} \subseteq I$ for any closed ideal $I \subset \mathcal{O}_X(U)$ such that $(I)_0 \subseteq Y \cap U$.

□

Universal property of open subspaces. *Let $i\colon U \hookrightarrow X$ be an open differentiable subspace of a differentiable space X. If T is a differentiable space, then the following natural map is bijective:*

$$\mathrm{Hom}\,(T,U) \longrightarrow \begin{bmatrix} \text{Morphisms } T \xrightarrow{\psi} X \\ \text{such that } \psi(T) \subseteq U \end{bmatrix} \quad , \quad \phi \mapsto i\phi\ .$$

Proof. Easy to check.

□

Universal property of closed subspaces. *Let $i\colon Y \hookrightarrow X$ be a closed differentiable subspace of a differentiable space X, defined by a sheaf of ideals $\mathcal{I}$. If T is a differentiable space, then the following natural map is bijective:*

$$\mathrm{Hom}\,(T,Y) \longrightarrow \begin{bmatrix} \text{Morphisms } T \xrightarrow{\psi} X \\ \text{such that } \psi^*\mathcal{I} = 0 \end{bmatrix} \quad , \quad \phi \mapsto i\phi\ .$$

Proof. Let $\psi\colon T \to X$ be a morphism such that $\psi^*\mathcal{I} = 0$. By hypothesis Y is the support of $\mathcal{O}_X/\mathcal{I}$, so that $\mathcal{I}_x = \mathcal{O}_{X,x}$ when $x \notin Y$. Hence the condition $\psi^*\mathcal{I} = 0$ implies that $\psi(T) \subseteq Y$, and there is a unique continuous map $\phi\colon T \to Y$ such that $\psi = i\phi$.

Since $\psi^*\mathcal{I} = 0$, the morphism of sheaves $\psi^*\colon \mathcal{O}_X \to \psi_*\mathcal{O}_T = i_*\phi_*(\mathcal{O}_T)$ factors through a morphism of sheaves $\mathcal{O}_X/\mathcal{I} \to i_*\phi_*(\mathcal{O}_T)$, which defines a morphism of sheaves $\phi^*\colon \mathcal{O}_Y = i^*(\mathcal{O}_X/\mathcal{I}) \to \phi_*\mathcal{O}_T$. Now it is easy to check that $(\phi,\phi^*)\colon (T,\mathcal{O}_T) \to (Y,\mathcal{O}_Y)$ is the unique morphism such that $\psi = i\phi$.

□

5.3 Infinitesimal Neighbourhoods

Definition. Let $(X, \mathcal{O}_X)$ be a differentiable space and let m_p be the sheaf of all differentiable functions vanishing at given point $p \in X$. If $r \geq 0$, then the sheaf $\mathcal{O}_X/m_p^{r+1}$ vanishes on $X \setminus \{p\}$ and its stalk $\mathcal{O}_{X,p}/\mathfrak{m}_p^{r+1}$ at p is a differentiable algebra by 2.26. Therefore

$$(\{p\}, \mathcal{O}_X/m_p^{r+1}) \; = \; \mathrm{Spec}_r\,(\mathcal{O}_{X,p}/\mathfrak{m}_p^{r+1})$$

is a differentiable space and we conclude that the sheaf of ideals m_p^{r+1} defines a closed differentiable subspace of X, called the r-th **infinitesimal neighbourhood** of p in X. We denote it by $U_p^r(X)$ or U_p^r:

$$U_p^r \; := \; \mathrm{Spec}_r\,\left(\mathcal{O}_{X,p}/m_p^{r+1}\right)\; .$$

The restriction of a differentiable function $f \in \mathcal{O}_X(X)$ to the subspace U_p^r is said to be the r**-th jet** of f at the point p, and we denote it by $j_p^r f$:

$$j_p^r f \; = \; f|_{U_p^r} \; = \; [f] \; \in \mathcal{O}_{X,p}/m_p^{r+1}\; .$$

When $r \leq s$, we have $m_p^{s+1} \subseteq m_p^{r+1}$, so that $U_p^r \subseteq U_p^s$:

$$p \; = \; U_p^0 \; \subseteq \; U_p^1 \subseteq \; U_p^2 \; \subseteq \; \ldots \; \subseteq \; U_p^r \; \subseteq \; \ldots \; \subseteq \; X\; .$$

Let $\varphi\colon Y \to X$ be a morphism of differentiable spaces and let $\varphi(q) = p$. Since $\varphi^*(\mathfrak{m}_p^{r+1}) \subseteq \mathfrak{m}_q^{r+1}$, the morphism $\varphi^*\colon \mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$ induces a morphism of $\mathbb{R}$-algebras $\varphi^*\colon \mathcal{O}_{X,p}/\mathfrak{m}_p^{r+1} \to \mathcal{O}_{Y,q}/\mathfrak{m}_q^{r+1}$, which defines a morphism of differentiable spaces $\varphi_q^r\colon U_q^r(Y) \to U_p^r(X)$ such that the following square is commutative:

$$\begin{array}{ccc} U_q^r(Y) & \hookrightarrow & Y \\ \Big\downarrow{\scriptstyle \varphi_q^r} & & \Big\downarrow{\scriptstyle \varphi} \\ U_p^r(X) & \hookrightarrow & X \end{array}$$

Infinitesimal neighbourhoods only depend on the ring of germs $\mathcal{O}_{X,p}$. If V is an open neighbourhood of p in X, then $\mathcal{O}_{X,p} = \mathcal{O}_{V,p}$, so that the r-th infinitesimal neighbourhood of p in V may be canonically identified with the r-th infinitesimal neighbourhood of p in X:

$$U_p^r(V) \; = \; U_p^r(X)\; .$$

In general, if $i\colon Y \hookrightarrow X$ is a differentiable subspace of a differentiable space X, then the restriction morphism $i^*\colon \mathcal{O}_{X,p} \to \mathcal{O}_{Y,p}$ is surjective at any point $p \in Y$; hence so is the induced morphism

$$\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{r+1} \;\longrightarrow\; \mathcal{O}_{Y,p}/\mathfrak{m}_{Y,p}^{r+1}$$

and the r-th infinitesimal neighbourhood of p in Y may be canonically identified with a closed differentiable subspace of the r-th infinitesimal neighbourhood of p in X:

$$Y \subseteq X \;\Rightarrow\; U_p^r(Y) \subseteq U_p^r(X)\; .$$

Proposition 5.11. *Let f be a differentiable function on a differentiable space X, i.e., $f \in \mathcal{O}_X(X)$. If f vanishes on any infinitesimal neighbourhood of every point of X, then $f = 0$:*

$$f = 0 \;\Leftrightarrow\; f|_{U_x^r} = 0\,,\; \forall\, x \in X,\, r \in \mathbb{N}\,.$$

Proof. It is a restatement of 2.17.

□

5.4 Infinitely Near Points

Definition. Let p be a point of a differentiable space $(X, \mathcal{O}_X)$ and let $\mathfrak{m}_p$ be the unique maximal ideal of $\mathcal{O}_{X,p}$. We say that the real vector space

$$T_pX \;=\; \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{X,p}, \mathcal{O}_{X,p}/\mathfrak{m}_p)$$

is the **tangent space** to X at the point p. Its dual vector space T_p^*X is said to be the **cotangent space** to X at p.

For any germ $f \in \mathcal{O}_{X,p}$, the **differential** of f at p is the 1–form $\mathrm{d}_pf \in T_p^*X$ defined by the formula

$$(\mathrm{d}_pf)(D) = Df\ .$$

Proposition 5.12. *There exists a canonical isomorphism*

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \;=\; T_p^*X \quad , \quad [f] \mapsto \mathrm{d}_pf$$

Proof. The proof of 1.14 remains valid in this new setting.

□

Let $\varphi\colon Y \to X$ be a morphism of differentiable spaces and let $q \in Y$. We put $p = \varphi(q)$. The morphism of sheaves $\varphi^*\colon \varphi^*\mathcal{O}_X \to \mathcal{O}_Y$ induces a morphism of rings $\varphi^*\colon \mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$; hence it induces linear maps

$$\begin{aligned} \varphi_* \;:\; T_qY \longrightarrow T_pX \qquad &, \qquad \varphi_*(D) = D \circ \varphi^*\ , \\ \varphi^* \;:\; T_p^*X \longrightarrow T_q^*Y \qquad &, \qquad \varphi^*(\omega) = \omega \circ \varphi_*\ . \end{aligned}$$

Proposition 5.13. *Let X be a differentiable space and let $i\colon Y \hookrightarrow X$ be a differentiable subspace. If $p \in Y$, then the tangent linear map $i_*\colon T_pY \to T_pX$ is injective. Therefore, $i^*\colon T_p^*X \to T_p^*Y$ is surjective.*

Proof. The restriction morphism $i^*\colon \mathcal{O}_{X,p} \to \mathcal{O}_{Y,p}$ is surjective; hence the linear map $i_*\colon \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{Y,p}, \mathbb{R}) \to \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{X,p}, \mathbb{R})$ is injective.

□

Proposition 5.14. *Let p be a point of a differentiable space X. The tangent linear map $T_p\left(U_p^r\right) \to T_pX$ is an isomorphism for any $r \geq 1$.*

Proof. Let $\overline{\mathfrak{m}}_p$ be the image of $\mathfrak{m}_p$ in $\mathcal{O}_{X,p}/\mathfrak{m}_p^{r+1}$, which is the algebra of all differentiable functions on the infinitesimal neighbourhood U_p^r. Since $r+1 \geq 2$ the natural map $T_p^*X = \mathfrak{m}_p/\mathfrak{m}_p^2 \to \overline{\mathfrak{m}}_p/\overline{\mathfrak{m}}_p^2 = T_p^*(U_p^r)$ is an isomorphism; hence so is its transpose linear map $T_p\left(U_p^r\right) \to T_pX$.

□

Definition. The real spectrum of the differentiable algebra $\mathbb{R}[\varepsilon] := \mathbb{R} \oplus \mathbb{R}\,\varepsilon$, where $\varepsilon^2 = 0$ (see 2.33), is said to be a **pair of infinitely near points**, and we denote it by

$$\mathbf{v}_p \ := \ \mathrm{Spec}_r\,\mathbb{R}[\varepsilon]\ .$$

Note that $\mathbb{R}[\varepsilon]$ has a unique maximal ideal (namely $\mathfrak{m} = \mathbb{R}\varepsilon$), which defines a point p of $\mathbf{v}_p$

$$p \hookrightarrow \mathbf{v}_p\ .$$

So any morphism $\psi\colon \mathbf{v}_p \to X$ into a differentiable space X induces, by restriction, a morphism $\psi|_p\colon p \to X$; that is to say, it defines a point $x \in X$, and we say that ψ is an **infinitely near point** to x in X.

Lemma 5.15. *Let $\mathcal{O}$ be an $\mathbb{R}$-algebra with a unique maximal ideal $\mathfrak{m}$. If $\mathcal{O}/\mathfrak{m} = \mathbb{R}$, then*

$$\mathrm{Hom}_{\mathbb{R}\text{-alg}}(\mathcal{O}, \mathbb{R}[\varepsilon]) \ = \ \mathrm{Der}_{\mathbb{R}}(\mathcal{O}, \mathcal{O}/\mathfrak{m})\ .$$

Proof. A map $\psi\colon \mathcal{O} \to \mathbb{R}[\varepsilon] = \mathbb{R} \oplus \mathbb{R}\,\varepsilon$, $\psi(f) = \pi(f) + D(f)\varepsilon$, is a morphism of $\mathbb{R}$-algebras if and only if so is $\pi\colon \mathcal{O} \to \mathbb{R} = \mathcal{O}/\mathfrak{m}$ (i.e., π is the canonical projection) and $D\colon \mathcal{O} \to \mathbb{R} = \mathcal{O}/\mathfrak{m}$ is an $\mathbb{R}$-derivation.

□

Proposition 5.16. *If p is a point of a differentiable space X, then*

$$T_pX \ = \ \begin{bmatrix}\text{Infinitely near}\\ \text{points to } p\end{bmatrix}\ .$$

Proof. Let $\mathcal{O} = \mathcal{O}_{X,p}$. Infinitely near points correspond with morphisms of $\mathbb{R}$-algebras $\phi\colon \mathcal{O} \to \mathbb{R}[\varepsilon]$ and, by 5.15, we have

$$\mathrm{Hom}_{\mathbb{R}\text{-alg}}(\mathcal{O}, \mathbb{R}[\varepsilon]) \ = \ \mathrm{Der}_{\mathbb{R}}(\mathcal{O}, \mathcal{O}/\mathfrak{m}_p) \ = \ T_pX\ .$$

□

Definition. Let p be a point of a differentiable space X. We say that some germs of differentiable functions $f_1, \ldots, f_n \in \mathcal{O}_{X,p}$ **separate infinitely near points** to p if the differentials $\mathrm{d}_p f_1, \ldots, \mathrm{d}_p f_n$ generate the cotangent space T_p^*X.

5.5 Local Embeddings

Definition. A morphism of differentiable spaces $\varphi\colon Y \to X$ is said to be an **embedding** if it admits some factorization $\varphi = i\phi$

$$Y \xrightarrow{\phi} Y' \xrightarrow{i} X\,,$$

where i is the inclusion morphism of a differentiable subspace of X and ϕ is an isomorphism. An embedding φ is said to be **open** or **closed** when so is the differentiable subspace Y' of X.

A morphism of differentiable spaces $\varphi : Y \to X$ is said to be a **local embedding** at a point $y \in Y$ if there exists an open neighbourhood U of y in Y such that $\varphi|_U\colon U \to X$ is an embedding.

By definition, a morphism $\varphi\colon Y \to X$ is an embedding when it induces a homeomorphism of Y onto a locally closed subspace of X and the morphism of sheaves $\varphi^*\colon \varphi^*\mathcal{O}_X \to \mathcal{O}_Y$ is surjective. An embedding $\varphi\colon Y \to X$ is closed if and only if $\varphi(Y)$ is a closed set in X.

According to 5.13, if $\varphi\colon Y \to X$ is an embedding, then the tangent linear map $\varphi_{*,y}\colon T_yY \to T_{\varphi(y)}X$ is injective at any point $y \in Y$.

By 5.8, affine differentiable spaces are just differentiable spaces which admit an embedding into some affine space $\mathbb{R}^n$. Moreover, 5.6 may be stated as follows:

Proposition 5.17. *Let X be an affine differentiable space. A morphism of differentiable spaces $(j, j^*)\colon Y \to X$ is a closed embedding if and only if Y is affine and $j^*\colon \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ is surjective.*

Lemma 5.18. *Let p be a point of a differentiable space X. There exists an open neighbourhood U of p and a closed embedding $j\colon U \hookrightarrow \mathbb{R}^m$ such that*

$$j_*\colon T_pU \longrightarrow T_{j(p)}\mathbb{R}^m$$

is an isomorphism.

Proof. We may assume that $X = \operatorname{Spec}_r \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$. Using the exact sequence

$$0 \longrightarrow \mathfrak{a}/\mathfrak{a} \cap \mathfrak{m}^2_{\mathbb{R}^n,p} \longrightarrow \mathfrak{m}_{\mathbb{R}^n,p}/\mathfrak{m}^2_{\mathbb{R}^n,p} \longrightarrow \mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p} \longrightarrow 0$$

and 5.12, it is easy to obtain smooth functions $(y_1, \ldots, y_m, u_{m+1}, \ldots, u_n)$ on $\mathbb{R}^n$ such that

$$\{\mathrm{d}_p y_1, \ldots, \mathrm{d}_p y_m, \mathrm{d}_p u_{m+1}, \ldots, \mathrm{d}_p u_n\}$$

is a basis of $T^*_p\mathbb{R}^n$, $\{\mathrm{d}_p y_1, \ldots, \mathrm{d}_p y_m\}$ is a basis of T^*_pX and $u_{m+1}, \ldots, u_n \in \mathfrak{a}$.

Hence $(y_1, \ldots, y_m, u_{m+1}, \ldots, u_n)$ is a local coordinate system of $\mathbb{R}^n$ at p and, in some open neighbourhood U' of p, the equations $u_{m+1} = 0,\ \ldots,\ u_n = 0$ define a smooth submanifold Y, which may be assumed to be diffeomorphic to $\mathbb{R}^m$. Now $U = U' \cap X$ is a closed differentiable subspace of Y because $(u_{m+1}, \ldots, u_n) \subseteq \mathfrak{a}$ and $\mathfrak{p}_Y = (u_{m+1}, \ldots, u_n)$ by 2.7. Finally the linear map $T_pU \hookrightarrow T_pY$ is injective by 5.13 and surjective because both vector spaces have the same dimension m.

□

Definition. The dimension of the real vector space T_pX is said to be the **embedding dimension** of X at the point p, since it is the least dimension of a smooth manifold where a neighbourhood of p may be embedded as a differentiable subspace.

Corollary 5.19. *If X is a differentiable space, then the embedding dimension $d(x) := \dim T_xX$ is an upper semicontinuous function on X. In particular, when X is compact, it is bounded.*

Proof. We have $d(x) \leq m = d(p)$ at any point x of the neighbourhood U considered in 5.18.

□

Corollary 5.20. *Let p be a point of a differentiable space X. If $m = \dim T_pX$, then p has some neighbourhood in X of topological dimension $\leq m$.*

Proof. By 5.18, p has a neighbourhood U which is homeomorphic to a closed set in $\mathbb{R}^m$; hence $\dim U \leq m$.

□

Theorem 5.21. *A morphism of differentiable spaces $\varphi\colon Y \longrightarrow X$ is a local embedding at a given point $y \in Y$ if and only if the tangent linear map $\varphi_{*,y}\colon T_yY \to T_{\varphi(y)}X$ is injective.*

Proof. Let $x = \varphi(y)$ and let us assume that $\varphi_*\colon T_yY \to T_xX$ is injective. We may assume that X and Y are closed differentiable subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and, by 5.18, that $T_yY = T_y\mathbb{R}^m$. According to 2.21 we have commutative diagrams

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{R}^m \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\phi} \\ X & \hookrightarrow & \mathbb{R}^n \end{array} \qquad\qquad \begin{array}{ccc} T_yY & = & T_y\mathbb{R}^m \\ \downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\phi_*} \\ T_xX & \hookrightarrow & T_x\mathbb{R}^n \end{array}$$

and, φ_* being injective, we obtain that ϕ_* is injective. Replacing $\mathbb{R}^n$ and $\mathbb{R}^m$ by some neighbourhoods U_x and V_y of x and y respectively, we may assume that $\phi\colon V_y \to U_x$ is a closed embedding. Compositions of closed embeddings are closed embeddings, so that the composition morphism $V_y \cap Y \to V_y \to U_x$ is a closed embedding. This morphism factors through $\varphi\colon V_y \cap Y \to U_x \cap X$ and the closed embedding $U_x \cap X \to U_x$, hence the first one also is a closed embedding.

The converse statement is clear from 5.13.

□

Corollary 5.22. *A morphism of differentiable spaces $\varphi : Y \to X$ is a local embedding at a given point $y \in Y$ if and only if its restriction $\varphi|_{U_y^1}$ to the first infinitesimal neighbourhood U_y^1 of y in Y is an embedding.*

Proof. According to 5.14, the inclusion morphism $i\colon U_y^1 \hookrightarrow Y$ induces a linear isomorphism $i_*\colon T_y\left(U_y^1\right) \simeq T_yY$. We conclude by 5.21, since the unique open neighbourhood of y in U_y^1 is just U_y^1.

□

Corollary 5.23. *Let $\varphi\colon Y \to X$ be a morphism of differentiable spaces and let $y \in Y$, $x = \varphi(y) \in X$. If the tangent linear map $\varphi_*\colon T_yY \to T_xX$ is injective, then the morphism of rings $\varphi^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is surjective.*

5.6 Embedding Theorem

Theorem 5.24. *A morphism of differentiable spaces $\varphi : Y \to X$ is an embedding if and only if it satisfies the following conditions:*

1. *$\varphi_* : T_yY \to T_{\varphi(y)}X$ is injective at any point $y \in Y$.*
2. *φ induces a homeomorphism of Y onto $\varphi(Y)$.*

Proof. Both conditions are obviously necessary.

Conversely, if φ satisfies both conditions, then, by 5.21, locally $\varphi(Y)$ is a locally closed subspace of X; hence it is locally closed in X. By 5.23 the morphism of sheaves $\varphi^*\colon \varphi^*\mathcal{O}_X \to \mathcal{O}_Y$ is surjective, hence $\varphi\colon Y \to X$ is an embedding.

□

Corollary 5.25. *Let X be a compact separated differentiable space. A morphism of differentiable spaces $f = (f_1, \ldots, f_n)\colon X \to \mathbb{R}^n$ is a closed embedding if and only if the differentiable functions $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ separate points and infinitely near points of X.*

Proof. Condition 5.24.1 states that the linear map

$$f^*\colon T^*_{f(x)}\mathbb{R}^n \longrightarrow T^*_xX \quad , \quad f^*(\mathrm{d}x_i) = \mathrm{d}f_i$$

is surjective at any point $x \in X$; hence it says that $f_1, \ldots, f_n$ separate infinitely near points.

On the other hand, $f_1, \ldots, f_n$ separate points of X just when the continuous map $f\colon X \to f(X)$ is bijective; hence this fact is equivalent to condition 5.24.2 when X is a separated compact space, because in such a case so is $f(X)$ and any continuous bijection between separated compact spaces is a homeomorphism.

Finally, any embedding $\varphi\colon X \hookrightarrow \mathbb{R}^n$ is closed because $\varphi(X)$ is closed in $\mathbb{R}^n$, since it is compact.

□

Corollary 5.26. *Let $f_1, \ldots, f_n$ be differentiable functions on an affine compact differentiable space $X = \mathrm{Spec}_r A$. The subalgebra of A generated by $f_1, \ldots, f_n$ is dense if and only if $f_1, \ldots, f_n$ separate points and infinitely near points of X.*

Proof. If the subalgebra $\mathbb{R}[f_1,\dots,f_n]$ is dense in A, then $f_1,\dots,f_n$ separate points and infinitely near points of X because we have continuous epimorphisms

$$A \longrightarrow A/(\mathfrak{m}_x \cap \mathfrak{m}_y) = \mathbb{R}\oplus\mathbb{R} \quad , \quad f \mapsto (f(x), f(y)) \ ,$$
$$A \longrightarrow A/\mathfrak{m}_x^2 = \mathbb{R}\oplus T_x^*X \quad , \quad f \mapsto (f(x), \mathrm{d}_x f) \ .$$

Conversely, if $f_1,\dots,f_n$ separate points and infinitely near points, then $(f_1,\dots,f_n)\colon X \to \mathbb{R}^n$ is a closed embedding by 5.25. Hence the corresponding morphism of $\mathbb{R}$-algebras

$$\mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow A \quad , \quad x_i \mapsto f_i \ ,$$

is surjective (by 5.17) and it is continuous (by 2.23). Since (see [39]) the ring of polynomials $\mathbb{R}[x_1,\dots,x_n]$ is dense in $\mathcal{C}^\infty(\mathbb{R}^n)$, we conclude that its image $\mathbb{R}[f_1,\dots,f_n]$ is dense in A.

□

Corollary 5.27. *Let $(X,\mathcal{O}_X)$ be a differentiable space. If Y is a topological subspace of X and $\mathcal{I}$ is a sheaf of ideals of $\mathcal{O}_X|_Y$ such that $(Y,\mathcal{O}_X/\mathcal{I})$ is a differentiable space, then Y is locally closed in X. Hence $(Y,\mathcal{O}_X/\mathcal{I})$ is a differentiable subspace of $(X,\mathcal{O}_X)$.*

Proof. Let $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$ and let us consider the morphism of differentiable spaces $(i,i^*)\colon (Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$, where $i\colon Y \hookrightarrow X$ is the inclusion map and $i^*\colon i^*\mathcal{O}_X = \mathcal{O}_X|_Y \to \mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$ is the canonical projection. For any $y\in Y$ the morphism $i^*\colon \mathcal{O}_{X,y} \to \mathcal{O}_{Y,y}$ is surjective, hence $i_{*,y}\colon T_yY \to T_yX$ is injective. Now 5.24 let us conclude that (i,i^*) is an embedding. In particular $Y = i(Y)$ is locally closed in X.

□

Embedding Theorem. *A differentiable space X is affine if and only if it is a separated space with bounded embedding dimension and its topology has a countable basis.*

Proof. Such conditions are obviously necessary.

Conversely, let m be an upper bound of the function $d(x) = \dim T_xX$. By 5.18, X may be covered by affine open subspaces which are isomorphic to subspaces of $\mathbb{R}^m$. By 5.20 and 4.13, it follows that X has a locally finite open cover (hence countable, since the topology of X has a countable basis) which is a finite union of some families of disjoint open subspaces, all of them isomorphic to subspaces of $\mathbb{R}^m$. Now, any disjoint countable union of subspaces of $\mathbb{R}^m$ is isomorphic to some subspace of $\mathbb{R}^m$, hence it is affine by 5.8 and we obtain the existence of a finite affine open cover $\{U_1,\dots,U_n\}$ of X.

For each index $i = 1,\dots,n$, there are differentiable functions $f_{i1},\dots,f_{is}$ on U_i defining a closed embedding $(f_{i1},\dots,f_{is})\colon U_i \to \mathbb{R}^s$. In particular such functions separate points of U_i and the corresponding linear map $T_xU_i \to T_x\mathbb{R}^s$ is injective at any point $x\in U_i$. Since X is paracompact, there are affine open covers $\{V_1,\dots,V_n\}$ and $\{W_1,\dots,W_n\}$ of X such that $\overline{W}_i \subseteq V_i$ and $\overline{V}_i \subseteq U_i$ for any index i. Since U_i is affine, there exists a differentiable function ϕ_i on U_i such that $\phi_i|_{W_i} = 1$ and $\phi_i|_{U_i - \overline{V}_i} = 0$, so that ϕ_i, $\phi_i f_{ij} \in \mathcal{O}_X(X)$, when extended by 0.

If a finite family of global differentiable functions $g_1, \dots, g_r$ contains all these functions ϕ_i and $\phi_i f_{ij}$, then the morphism $\varphi = (g_1, \dots, g_r) \colon X \to \mathbb{R}^r$ separates points of X and the tangent linear map $\varphi_* \colon T_x X \to T_{\varphi(x)}\mathbb{R}^r$ is injective at any point $x \in X$.

We only have to prove that the functions ϕ_i, $\phi_i f_{ij}$ separate points of X. If they coincide at two points $x, y \in X$, let i be an index such that $x \in W_i$; then $\phi_i(y) = \phi_i(x) = 1$ and $f_{ij}(y) = f_{ij}(x)$ for any index j. It follows that $y \in U_i$; so that $x = y$ since the functions f_{ij} separate points of U_i.

If we may choose this family $(g_1, \dots, g_r)$ so that $\varphi \colon X \to \mathbb{R}^r$ defines a homeomorphism of X onto a closed subset of $\mathbb{R}^r$, then 5.24 let us conclude that φ is a closed embedding: X is affine. Let $\phi \colon X \to \mathbb{R}$ be a proper differentiable function (it exists by 4.5). If $g_1, \dots, g_r$ separate points of X and $g_1 = \phi$, then the continuous map $\varphi \colon X \to \mathbb{R}^r$ is closed and injective; therefore it defines a homeomorphism of X onto a closed set in $\mathbb{R}^r$. Resuming, $\varphi = (\phi, \phi_i, \phi_i f_{ij})$ defines a closed embedding $X \hookrightarrow \mathbb{R}^r$.

□

Corollary 5.28. *Any compact separated differentiable space X is affine.*

Proof. The function $d(x) = \dim T_x X$ is bounded on X and the topology of X has a countable basis, because X has a finite affine open cover. We conclude by the Embedding theorem.

□

Corollary 5.29. *A smooth manifold $\mathcal{V}$ is affine if and only if it is separated, finite-dimensional, and its topology has a countable basis.*

Proof. We have $d(x) = \dim T_x\mathcal{V} = \dim_x \mathcal{V} \leq \dim \mathcal{V}$ at any point $x \in \mathcal{V}$.

□

Corollary 5.30. *Let $\mathcal{V}$ be a separated finite-dimensional smooth manifold whose topology has a countable basis. The module of all $\mathcal{C}^\infty$-differentiable tensor fields of type (p, q) on $\mathcal{V}$ is a finitely generated projective $\mathcal{C}^\infty(\mathcal{V})$-module.*

Proof. Since $\mathcal{V}$ is affine we have $(\mathcal{V}, \mathcal{C}^\infty_{\mathcal{V}}) = \operatorname{Spec}_r \mathcal{C}^\infty(\mathcal{V})$. The sheaf $\mathcal{T}_p^q$ of $\mathcal{C}^\infty$-differentiable tensor fields of type (p, q) is a locally free $\mathcal{C}^\infty_{\mathcal{V}}$-module of bounded rank. Now 4.16 let us conclude that $\mathcal{T}_p^q(\mathcal{V})$ is a finitely generated projective $\mathcal{C}^\infty(\mathcal{V})$-module.

□

6 Topological Tensor Products

Let $k \to A_1$ and $k \to A_2$ be morphisms of differentiable algebras. Generally, $A_1 \otimes_k A_2$ may not be a differentiable algebra (the tensor topology of $A_1 \otimes_k A_2$ may not be complete). Let us denote by $A_1 \widehat{\otimes}_k A_2$ the completion of $A_1 \otimes_k A_2$. We shall prove that $A_1 \widehat{\otimes}_k A_2$ is a differentiable algebra and that it has the universal property of a coproduct:

$$\mathrm{Hom}_{k\text{-alg}}(A_1 \widehat{\otimes}_k A_2, B) \;=\; \mathrm{Hom}_{k\text{-alg}}(A_1, B) \times \mathrm{Hom}_{k\text{-alg}}(A_2, B)$$

for any morphism $k \to B$ of differentiable algebras. This result will be the main ingredient for the construction of fibred products of differentiable spaces.

6.1 Locally Convex Modules

Let us recall that a locally m-convex algebra is defined to be an $\mathbb{R}$-algebra (commutative with unity) A endowed with a topology defined by a family $\{q_i\}$ of submultiplicative seminorms: $q_i(ab) \leq q_i(a)q_i(b)$.

If I is an ideal of a locally convex m-algebra A, then A/I is a locally m-convex algebra with the quotient topology: If $\{q_i\}$ is a fundamental system of submultiplicative seminorms of A, then the topology of A/I is defined by the submultiplicative seminorms $\overline{q}_i([a]) = \inf_{b \in I} q_i(a+b)$.

The canonical projection $\pi\colon A \to A/I$ is an open map.

The closure $\overline{I}$ of an ideal I is again an ideal of A.

Morphisms of locally m-convex algebras are defined to be continuous morphisms of $\mathbb{R}$-algebras. Locally m-convex algebras, with continuous morphisms of algebras, define a category. The set of all morphisms of locally m-convex algebras $A \to B$ is denoted by $\mathrm{Hom}_{m\text{-alg}}(A, B)$.

We say that a locally m-convex algebra is complete when so it is as a locally convex space (hence it is separated by definition). The completion $\widehat{A}$ of a locally m-convex algebra A is a locally m-convex algebra, and it has a universal property: Any morphism of locally m-convex algebras $A \to C$, where C is complete, factors in a unique way through the completion:

$$\mathrm{Hom}_{m\text{-alg}}(\widehat{A}, C) \;=\; \mathrm{Hom}_{m\text{-alg}}(A, C)\ .$$

J.A. Navarro González and J.B. Sancho de Salas: LNM 1824, pp. 69–77, 2003.

Let us recall that a locally m-convex algebra is said to be a Fréchet algebra if it is metrizable and complete.

If I is a closed ideal of a Fréchet algebra A, then A/I is a Fréchet algebra. Recall that any differentiable algebra, with the canonical topology, is a Fréchet algebra (see 2.23).

Definition. Let A be a locally m-convex algebra. A **locally convex A-module** is defined to be any A-module M endowed with a locally convex topology such that the map $A \times M \to M$, $(a, m) \mapsto am$, is continuous.

If N is a submodule of a locally convex A-module M, then the quotient topology defines on M/N a structure of locally convex A-module, because the following square

$$\begin{array}{ccc} A \times M & \xrightarrow{\cdot} & M \\ \downarrow & & \downarrow \\ A \times (M/N) & \xrightarrow{\cdot} & M/N \end{array}$$

is commutative and $A \times M \to A \times (M/N)$ is an open map. A similar argument shows that M/IM is a locally convex (A/I)-module for any ideal I of A.

Morphisms of locally convex A-modules are defined to be continuous morphisms of A-modules. Locally convex A-modules, with continuous morphisms of A-modules, define a category. The A-module of all morphisms of locally convex A-modules $M \to N$ is denoted by $\operatorname{Hom}_A(M, N)$.

A locally convex A-module is said to be **complete** when so it is as a locally convex vector space. The completion $\widehat{M}$ of a locally convex A-module M is a complete locally convex $\widehat{A}$-module (hence a complete locally convex A-module). For any complete locally convex $\widehat{A}$-module N, we have:

$$\operatorname{Hom}_{\widehat{A}}(\widehat{M}, N) \;=\; \operatorname{Hom}_A(\widehat{M}, N) \;=\; \operatorname{Hom}_A(M, N)\;.$$

A locally convex A-module is said to be a **Fréchet A-module** when so it is as a locally convex vector space. For example, if $\mathcal{V}$ is an affine smooth manifold, then the $\mathcal{C}^\infty(\mathcal{V})$-module $\mathcal{T}_p^q(\mathcal{V})$ of all $\mathcal{C}^\infty$-differentiable tensor fields of type (p, q) on $\mathcal{V}$, with the topology of the uniform convergence on compact sets of the components and their derivatives, is a Fréchet module.

Note that if N is a closed submodule of a Fréchet A-module M, then N and M/N also are Fréchet A-modules.

Definition. A sequence of morphisms of locally convex A-modules

$$M_1 \xrightarrow{j} M \xrightarrow{p} M_2 \longrightarrow 0$$

is said to be a **cokernel** if p is a surjective open morphism and $\operatorname{Im} j$ is a dense subspace of $\operatorname{Ker} p$.

From a rigorous categorial point of view, the previous definition provides the correct notion of cokernel when the category of *separated* locally convex A-modules is considered. In the category of general locally convex A-modules, the notion of cokernel might be different, but we maintain the above definition for the sake of simplicity, since in most of the situations the modules under consideration are always separated.

Proposition 6.1. *Let E be a locally convex vector space such that $\widehat{E}$ is Fréchet.*

(a) *If V is a vector subspace of E, then $\widehat{E/V} = \widehat{E}/\widehat{V}$.*

(b) *Completion preserves cokernels: If we have a cokernel of locally convex vector spaces*

$$E_1 \xrightarrow{j} E \xrightarrow{p} E_2 \longrightarrow 0 ,$$

then the sequence

$$\widehat{E}_1 \xrightarrow{\widehat{j}} \widehat{E} \xrightarrow{\widehat{p}} \widehat{E}_2 \longrightarrow 0$$

also is a cokernel.

Proof. (a) It is clear that $\widehat{V}$ is a closed vector subspace of $\widehat{E}$. On the other hand, for any complete locally convex vector space F, continuous linear maps $\widehat{E}/\widehat{V} \to F$ corresponds with continuous linear maps $E \to F$ vanishing on V; i.e., with continuous linear maps $E/V \to F$. Since $\widehat{E}/\widehat{V}$ is complete because $\widehat{E}$ is Fréchet, we conclude that

$$\widehat{E/V} = \widehat{E}/\widehat{V} .$$

(b) Let $K = \operatorname{Ker} p$, so that $E/K = E_2$. By (a) we have $\widehat{E}/\widehat{K} = \widehat{E}_2$, hence the natural projection $\widehat{p}\colon \widehat{E} \to \widehat{E}_2 = \widehat{E}/\widehat{K}$ is a surjective open map. Since $\operatorname{Im} j$ is dense in K and K is dense in $\widehat{K} = \operatorname{Ker}\widehat{p}$, we conclude that $\operatorname{Im}\widehat{j}$ is dense in $\operatorname{Ker}\widehat{p}$.

□

6.2 Tensor Product of Modules

Let $E_1, \dots, E_n$ be locally convex vector spaces. We shall always consider on $E_1 \otimes_{\mathbb{R}} \dots \otimes_{\mathbb{R}} E_n$ the finer locally convex topology ([16] 1.1) such that the canonical multilinear map $\otimes\colon E_1 \times \dots \times E_n \to E_1 \otimes_{\mathbb{R}} \dots \otimes_{\mathbb{R}} E_n$ is continuous; hence, if F is a locally convex vector space and $T\colon E_1 \times \dots \times E_n \to F$ is a continuous multilinear map, then there exists a unique continuous linear map $t\colon E_1 \otimes_{\mathbb{R}} \dots \otimes_{\mathbb{R}} E_n \to F$ such that $t(e_1 \otimes \dots \otimes e_n) = T(e_1, \dots, e_n)$.

The completion of $E_1 \otimes_{\mathbb{R}} \dots \otimes_{\mathbb{R}} E_n$ will be denoted by $E_1 \widehat{\otimes}_{\mathbb{R}} \dots \widehat{\otimes}_{\mathbb{R}} E_n$ and it has the corresponding universal property with respect to continuous multilinear maps into complete locally convex vector spaces.

The topology of $E_1 \otimes_{\mathbb{R}} \dots \otimes_{\mathbb{R}} E_n$ is defined by the seminorms $q_{i_1} \otimes \dots \otimes q_{i_n}$:

$$(q_{i_1} \otimes \dots \otimes q_{i_n})(u) \ = \ \inf_{u=\sum e_{j_1} \otimes \dots \otimes e_{j_n}} \{\textstyle\sum q_{i_1}(e_{j_1}) \cdots q_{i_n}(e_{j_n})\} ,$$

where $\{q_{i_1}\}, \dots, \{q_{i_n}\}$ are fundamental systems of seminorms on $E_1, \dots, E_n$ respectively. Moreover $(q_{i_1} \otimes \dots \otimes q_{i_n})(e_{j_1} \otimes \dots \otimes e_{j_n}) = q_{i_1}(e_{j_1}) \cdots q_{i_n}(e_{j_n})$. Using these seminorms it is easy to check that

$$(E_1 \otimes_{\mathbb{R}} E_2) \otimes_{\mathbb{R}} E_3 = E_1 \otimes_{\mathbb{R}} E_2 \otimes_{\mathbb{R}} E_3 \ .$$

If E_1, E_2 are separated vector spaces, then so is $E_1 \otimes_{\mathbb{R}} E_2$. If E_1, E_2 are Fréchet vector spaces, then so is $E_1 \widehat{\otimes}_{\mathbb{R}} E_2$.

Lemma 6.2. (a) *For any locally convex vector spaces $V \subseteq E$, F we have*

$$(E/V) \otimes_{\mathbb{R}} F \ = \ (E \otimes_{\mathbb{R}} F)/(V \otimes_{\mathbb{R}} F) \ .$$

(b) *If $h_1 \colon E_1 \to F_1$, $h_2 \colon E_2 \to F_2$ are open epimorphisms of locally convex vector spaces, then so is $h_1 \otimes h_2 \colon E_1 \otimes_{\mathbb{R}} E_2 \to F_1 \otimes_{\mathbb{R}} F_2$.*

Proof. (a) It is a direct consequence of the universal property of tensor products, because $E \times F \to (E/V) \times F$ is an open map.

(b) Since $F_1 = E_1/\operatorname{Ker} h_1$, we have $F_1 \otimes_{\mathbb{R}} E_2 = (E_1 \otimes_{\mathbb{R}} E_2)/(\operatorname{Ker} h_1 \otimes_{\mathbb{R}} E_2)$ by (a), i.e., $h_1 \otimes 1 \colon E_1 \otimes_{\mathbb{R}} E_2 \longrightarrow F_1 \otimes_{\mathbb{R}} E_2$ is an open epimorphism. Analogously, $1 \otimes h_2 \colon F_1 \otimes_{\mathbb{R}} E_2 \longrightarrow F_1 \otimes_{\mathbb{R}} F_2$ is an open epimorphism, hence so is the composition map $E_1 \otimes_{\mathbb{R}} E_2 \longrightarrow F_1 \otimes_{\mathbb{R}} E_2 \longrightarrow F_1 \otimes_{\mathbb{R}} F_2$.

□

Definition. Let A be a locally m-convex algebra and let M, N be locally convex A-modules. We shall always consider on $M \otimes_A N$ the final topology of the canonical epimorphism $M \otimes_{\mathbb{R}} N \to M \otimes_A N$. Its completion will be denoted by $M \widehat{\otimes}_A N$.

$M \otimes_A N$ is a locally convex A-module (hence $M \widehat{\otimes}_A N$ is a locally convex $\widehat{A}$-module). In fact, the trilinear map $A \times M \times N \to M \times N \to M \otimes_A N$, $(a, m, n) \mapsto (am, n) \mapsto am \otimes n$, is continuous; hence we have a continuous map $A \otimes_{\mathbb{R}} M \otimes_{\mathbb{R}} N \longrightarrow M \otimes_A N$. Then we have continuous morphisms:

$$A \times (M \otimes_{\mathbb{R}} N) \ \longrightarrow \ A \otimes_{\mathbb{R}} (M \otimes_{\mathbb{R}} N) = A \otimes_{\mathbb{R}} M \otimes_{\mathbb{R}} N \ \longrightarrow \ M \otimes_A N \ .$$

Since the natural map $A \times (M \otimes_{\mathbb{R}} N) \to A \times (M \otimes_A N)$ is open, we conclude that the product $A \times (M \otimes_A N) \to M \otimes_A N$ is continuous.

The canonical A-bilinear map $M \times N \xrightarrow{\otimes} M \otimes_A N$ is continuous and, for any locally convex A-module P, we have a natural bijection

$$\operatorname{Hom}_A(M \otimes_A N, P) \ = \ \operatorname{Bil}_A(M, N; P)$$

where $\operatorname{Bil}_A(M, N; P)$ denotes the A-module of all continuous A-bilinear maps $M \times N \to P$. Analogously, for any complete locally convex A-module P we have

$$\operatorname{Hom}_{\widehat{A}}(M \widehat{\otimes}_A N, P) \ = \ \operatorname{Hom}_A(M \widehat{\otimes}_A N, P) \ = \ \operatorname{Bil}_A(M, N; P) \ .$$

If $h_1 \colon M_1 \to N_1$, $h_2 \colon M_2 \to N_2$ are morphisms of locally convex A-modules, then the A-bilinear map $M_1 \times M_2 \to N_1 \otimes_A N_2$, $(m_1, m_2) \mapsto h_1(m_1) \otimes h_2(m_2)$, is continuous and it induces a morphism $h_1 \otimes h_2 \colon M_1 \otimes_A M_2 \to N_1 \otimes_A N_2$ of locally convex A-modules.

Lemma 6.3. *Let $h_1\colon M_1 \to N_1$ and $h_2\colon M_2 \to N_2$ be open epimorphisms of A-modules. Then*

(a) *$h_1 \otimes h_2\colon M_1 \otimes_A M_2 \to N_1 \otimes_A N_2$ is an open epimorphism.*

(b) *If $M_1 \widehat{\otimes}_A M_2$ is Fréchet, then $h_1 \widehat{\otimes} h_2\colon M_1 \widehat{\otimes}_A M_2 \to N_1 \widehat{\otimes}_A N_2$ is an open epimorphism.*

Proof. (a) It is a consequence of 6.2.b.

(b) It is a consequence of (a) and 6.1.b.

□

Proposition 6.4. *Let A be a locally m-convex algebra and let N be a locally convex A-module. The functor $\otimes_A N$ preserves cokernels: If*

$$M_1 \xrightarrow{\ j\ } M \xrightarrow{\ p\ } M_2 \longrightarrow 0$$

is a cokernel of locally convex A-modules, then the sequence

$$M_1 \otimes_A N \xrightarrow{\ j\otimes 1\ } M \otimes_A N \xrightarrow{\ p\otimes 1\ } M_2 \otimes_A N \longrightarrow 0$$

is a cokernel.

Proof. It is clear that $\operatorname{Im}(j \otimes 1)$ is dense in $\operatorname{Ker}(p \otimes 1)$, and $p \otimes 1$ is open by 6.3.

□

Proposition 6.5. *Let A be a locally m-convex algebra and let N be a Fréchet A-module. The functor $\widehat{\otimes}_A N$ preserves cokernels in the category of Fréchet A-modules: If*

$$M_1 \xrightarrow{\ j\ } M \xrightarrow{\ p\ } M_2 \longrightarrow 0$$

is a cokernel of Fréchet A-modules, then the sequence

$$M_1 \widehat{\otimes}_A N \xrightarrow{\ j\widehat{\otimes} 1\ } M \widehat{\otimes}_A N \xrightarrow{\ p\widehat{\otimes} 1\ } M_2 \widehat{\otimes}_A N \longrightarrow 0$$

is a cokernel of Fréchet A-modules.

Proof. It is a consequence of 6.4 and 6.1.b.

□

Remark. If $M_1 \to M \to M_2 \to 0$ is an exact sequence of Fréchet A-modules, it may be that the sequence $M_1 \widehat{\otimes}_A N \to M \widehat{\otimes}_A N \to M_2 \widehat{\otimes}_A N \to 0$ is not exact. For example, let $f(x)$ be a differentiable function such that the Taylor expansion of $f(x)$ at $x = 0$ is identically 0 and $f(x)$ has no other zero. Let us consider in $\mathcal{C}^\infty(\mathbb{R}^2)$ the ideals $\mathfrak{a} = (y + f(x))$ and $\mathfrak{b} = (y)$. By 2.7, $\mathfrak{a}$ and $\mathfrak{b}$ are the ideals of all differentiable functions vanishing on the submanifolds $y + f(x) = 0$ and $y = 0$, respectively. Hence $\mathfrak{a}$ and $\mathfrak{b}$ are closed ideals. The sequence

$$0 \longrightarrow \mathcal{C}^\infty(\mathbb{R}^2) \xrightarrow{\ y+f(x)\ } \mathcal{C}^\infty(\mathbb{R}^2) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^2)/\mathfrak{a} \longrightarrow 0$$

is exact, and applying $\widehat{\otimes}_{\mathcal{C}^\infty(\mathbb{R}^2)}\mathcal{C}^\infty(\mathbb{R}^2)/\mathfrak{b}$, we obtain the sequence

$$\mathcal{C}^\infty(\mathbb{R}) \xrightarrow{\cdot f(x)} \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R})/\overline{(f)} \longrightarrow 0 \,,$$

which is not exact because the ideal (f) is not closed in $\mathcal{C}^\infty(\mathbb{R})$. In fact, according to Whitney's spectral theorem, its closure is just the Whitney ideal W of the point $x=0$, which is not a finitely generated ideal, since we have $xW = W$ and otherwise Nakayama's lemma would imply that any element of W has null germ at $x=0$, an obviously false statement.

Proposition 6.6. $M\widehat{\otimes}_A N \;=\; \widehat{M}\,\widehat{\otimes}_A\widehat{N} \;=\; \widehat{M}\,\widehat{\otimes}_{\widehat{A}}\widehat{N}$

Proof. Let P be a complete locally convex A-module (hence a complete locally convex $\widehat{A}$-module). Any continuous A-bilinear map $M\times N \to P$ may be extended, in a unique way, so as to obtain a continuous A-bilinear map $\widehat{M}\times\widehat{N} \to P$, which is $\widehat{A}$-bilinear, i.e.,

$$\mathrm{Bil}_A(M,N;P) \;=\; \mathrm{Bil}_A(\widehat{M},\widehat{N};P) \;=\; \mathrm{Bil}_{\widehat{A}}(\widehat{M},\widehat{N};P)\,,$$
$$\mathrm{Hom}_A(M\widehat{\otimes}_A N,P) \;=\; \mathrm{Hom}_A(\widehat{M}\,\widehat{\otimes}_A\widehat{N},P) \;=\; \mathrm{Hom}_{\widehat{A}}(\widehat{M}\,\widehat{\otimes}_{\widehat{A}}\widehat{N},P)\,.$$

□

Proposition 6.7. *We have natural isomorphisms of locally convex A-modules*

$$\begin{aligned}(M\otimes_A N)\otimes_A P \;&=\; M\otimes_A(N\otimes_A P)\\ M\otimes_A N \;&=\; N\otimes_A M\,,\\ M\otimes_A(A/I) \;&=\; M/IM\,,\\ (A^r)\otimes_A M \;&=\; M^r\,.\end{aligned}$$

Proof. The natural continuous linear maps $M\otimes_{\mathbb{R}}(N\otimes_{\mathbb{R}}P)\to M\otimes_A(N\otimes_A P)$ and $(M\otimes_{\mathbb{R}}N)\otimes_{\mathbb{R}}P\to(M\otimes_A N)\otimes_A P$ are open and, by means of the natural topological isomorphism $M\otimes_{\mathbb{R}}(N\otimes_{\mathbb{R}}P)=(M\otimes_{\mathbb{R}}N)\otimes_{\mathbb{R}}P$, both have the same cokernel. Hence $(M\otimes_A N)\otimes_A P \;=\; M\otimes_A(N\otimes_A P)$.

The commutative property is obvious.

The morphism of A-modules $M\to M\times A/I\to M\otimes_A(A/I)$, $m\mapsto m\otimes 1$, is continuous and it vanishes on IM; hence it induces a continuous morphism of A-modules $M/IM \to M\otimes_A(A/I)$, $[m]\mapsto m\otimes 1$. Since $M\times A\to M$, $(m,a)\mapsto am$, is continuous we have that $M\times(A/I)\to M/IM$, $(m,[a])\mapsto[am]$, is an A-bilinear continuous map and then $M\otimes_A(A/I)\to M/IM$, $m\otimes[a]\mapsto[am]$ also is continuous. We conclude that $M/IM=M\otimes_A(A/I)$.

The morphism of A-modules $M^r\to(\bigoplus_i Ae_i)\otimes_A M$, $(m_i)\mapsto\sum_i e_i\otimes m_i$, is continuous because so it is on each summand, and its inverse $(A^r)\otimes_A M\to M^r$, $(a_i)\otimes m\mapsto(a_i m)$, is continuous.

□

Corollary 6.8. *We have natural isomorphisms of locally convex A-modules*

$$\begin{aligned}(M\,\widehat{\otimes}_A\,N)\widehat{\otimes}_A\,P &= M\,\widehat{\otimes}_A\,(N\,\widehat{\otimes}_A\,P)\ ,\\ M\,\widehat{\otimes}_A\,N &= N\,\widehat{\otimes}_A\,M\ ,\\ M\,\widehat{\otimes}_A\,(A/I) &= \widehat{M/IM}\ ,\\ (A^r)\,\widehat{\otimes}_A\,M &= \widehat{M}^r\ .\end{aligned}$$

In particular, we have $(A^r)\widehat{\otimes}_A M = M^r$ when M is a complete A-module and we have $M\widehat{\otimes}_A(A/I) = \widehat{M}/\widehat{IM}$ when $\widehat{M}$ is a Fréchet A-module.

Proof. It is a consequence of 6.7 and 6.6.

□

6.3 Base Change of Modules

Let $A \to B$ a morphism of locally m-convex algebras. It defines on B a structure of locally convex A-module. If M is a locally convex A-module, then $M \otimes_A B$ is a locally convex B-module. In fact, the trilinear map

$$M \times B \times B \longrightarrow M \times B \to M \otimes_A B \quad , \quad (m, b', b) \longmapsto m \otimes b'b\ ,$$

is continuous; hence we have continuous maps

$$(M \otimes_{\mathbb{R}} B) \times B \longrightarrow M \otimes_{\mathbb{R}} B \otimes_{\mathbb{R}} B \longrightarrow M \otimes_A B\ .$$

Since the natural map $(M \otimes_{\mathbb{R}} B) \times B \to (M \otimes_A B) \times B$ is open, we conclude that the product $(M \otimes_A B) \times B \to M \otimes_A B$ is continuous.

The morphism of A-modules $j \colon M \to M \otimes_A B$, $j(m) = m \otimes 1$, is continuous and, for any locally convex B-module N, we have a bijection

$$\mathrm{Hom}_B(M \otimes_A B, N) = \mathrm{Hom}_A(M, N) \quad , \quad h \longmapsto h \circ j\ .$$

Analogously, for any complete locally convex B-module N we have

$$\mathrm{Hom}_{\widehat{B}}(M\widehat{\otimes}_A B, N) = \mathrm{Hom}_A(M, N)\ .$$

Proposition 6.9. *Let $A \to B$ be a morphism of locally m-convex algebras, M a locally convex A-module and N a locally convex B-module. We have isomorphisms of locally convex B-modules*

$$\begin{aligned}(M \otimes_A B) \otimes_B N &= M \otimes_A N\ ,\\ (M\,\widehat{\otimes}_A\,B)\,\widehat{\otimes}_B\,N &= M\,\widehat{\otimes}_A\,N\ .\end{aligned}$$

Proof. The natural morphism of B-modules $(M \otimes_A B) \otimes_B N \to M \otimes_A N$ is continuous because the linear map $(M \otimes_{\mathbb{R}} B) \otimes_{\mathbb{R}} N \to (M \otimes_A B) \otimes_B N$ is open and the linear map $(M \otimes_{\mathbb{R}} B) \otimes_{\mathbb{R}} N \to M \otimes_A N$, $(m \otimes b) \otimes n \mapsto m \otimes bn$, is continuous. The inverse morphism $M \otimes_A N \to (M \otimes_A B) \otimes_B N$, $m \otimes n \mapsto (m \otimes 1) \otimes n$, is clearly continuous.

We conclude by 6.6.

□

Corollary 6.10. *Base changes are transitive and preserve tensor products:*

$$(M \otimes_A B) \otimes_B C = M \otimes_A C \ , \ (M \otimes_A N) \otimes_A B = (M \otimes_A B) \otimes_B (N \otimes_A B) \ ,$$
$$(M \widehat{\otimes}_A B) \widehat{\otimes}_B C = M \widehat{\otimes}_A C \ , \ (M \widehat{\otimes}_A N) \widehat{\otimes}_A B = (M \widehat{\otimes}_A B) \widehat{\otimes}_B (N \widehat{\otimes}_A B) \ .$$

Proposition 6.11. *Let A be a Fréchet algebra.*

(a) *If M is a finitely generated Fréchet A-module and N is a locally convex A-module, then any morphism of A-modules $M \to N$ is continuous.*

(b) *Any finitely generated projective A-module P admits a unique topology of Fréchet A-module. Moreover, if $A \to B$ is a morphism of Fréchet algebras, then $P \otimes_A B$ is a Fréchet B-module. In particular:*

$$P \otimes_A B \ = \ P \widehat{\otimes}_A B \ .$$

Proof. (a) Let us consider a continuous epimorphism $p\colon A^n \to M \to 0$ defined by some finite generating system of M. It is an open map, because A^n and M are Fréchet. Since any morphism of A-modules $A^n \to N$ is clearly continuous, we conclude that any morphism of A-modules $M \to N$ is continuous.

(b) If P is a finitely generated projective A-module, then $A^n \simeq P \oplus P'$, so that P is a Fréchet A-module with the topology induced by A^n, because P' is separated and the natural projection $A^n \to P'$ is continuous by (a). Such Fréchet topology is unique, since we have proved that it must coincide with the final topology of any epimorphism of A-modules $p\colon A^n \to P$. Moreover, p admits some A-linear section $s\colon P \to A^n$, which is continuous. Hence the open epimorphism $p \otimes Id\colon B^n = (A^n) \otimes_A B \to P \otimes_A B$ has the continuous section $s \otimes Id$, and we conclude that $P \otimes_A B$ is separated, hence it is Fréchet.

□

6.4 Tensor Product of Algebras

The tensor product $A \otimes_{\mathbb{R}} B$ of two locally m-convex algebras A, B is a locally m-convex algebra, since the seminorms $q_1 \otimes q_2$ are submultiplicative when so are the factors q_1, q_2. If $j_1 \colon k \to A$ and $j_2 \colon k \to B$ are morphisms of locally m-convex algebras, then $A \otimes_k B$ is a locally m-convex algebra and the morphisms $i_1 \colon A \to A \otimes_k B$, $i_1(a) = a \otimes 1$, and $i_2 \colon B \to A \otimes_k B$, $i_2(b) = 1 \otimes b$, are continuous. For any locally m-convex algebra C we have a bijection

$$\begin{aligned}\mathrm{Hom}_{m\text{-alg}}(A\otimes_k B, C) &= \mathrm{Hom}_{m\text{-alg}}(A,C)\times_{\mathrm{Hom}_{m\text{-alg}}(k,C)}\mathrm{Hom}_{m\text{-alg}}(B,C)\\ \varphi &\mapsto (\varphi\, i_1, \varphi\, i_2)\end{aligned}$$

That is to say, $(A\otimes_k B, i_1, i_2)$ is the coproduct of the morphisms j_1, j_2 in the category of locally m-convex algebras. In fact, given morphisms of locally m-convex algebras $h_1\colon A\to C$, $h_2\colon B\to C$ such that $h_1 j_1 = h_2 j_2$, the k-bilinear map $A\times B\to C$, $(a,b)\mapsto h_1(a)h_2(b)$, is continuous and it induces a continuous morphism of $\mathbb{R}$-algebras $h\colon A\otimes_k B\to C$ such that $hi_1 = h_1$ and $hi_2 = h_2$.

The completion $A\widehat{\otimes}_k B$ is a complete locally m-convex algebra endowed with morphisms $i_1\colon A\to A\widehat{\otimes}_k B$, $i_2\colon B\to A\widehat{\otimes}_k B$. When the algebras k, A, B are complete, $(A\widehat{\otimes}_k B, i_1, i_2)$ is the coproduct of j_1, j_2 in the category of complete locally m-convex algebras: For any complete locally m-convex algebra C we have

$$\mathrm{Hom}_{m\text{-alg}}(A\widehat{\otimes}_k B, C) \;=\; \mathrm{Hom}_{m\text{-alg}}(A,C)\times_{\mathrm{Hom}_{m\text{-alg}}(k,C)}\mathrm{Hom}_{m\text{-alg}}(B,C)\ .$$

Theorem 6.12. *There exists a natural isomorphism of Fréchet algebras*

$$\mathcal{C}^\infty(\mathbb{R}^n)\,\widehat{\otimes}_{\mathbb{R}}\,\mathcal{C}^\infty(\mathbb{R}^m) \;=\; \mathcal{C}^\infty(\mathbb{R}^{n+m})\ .$$

Proof. See [72] Th. 51.6.

□

Theorem 6.13. *If $k\to A$ and $k\to B$ are morphisms of differentiable algebras, then $A\widehat{\otimes}_k B$ is a differentiable algebra.*

Therefore, the category of differentiable algebras has finite coproducts.

Proof. If $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ and $B = \mathcal{C}^\infty(\mathbb{R}^m)/\mathfrak{b}$, then we have open epimorphisms

$$\mathcal{C}^\infty(\mathbb{R}^n)\otimes_{\mathbb{R}}\mathcal{C}^\infty(\mathbb{R}^m) \;\longrightarrow\; A\otimes_{\mathbb{R}} B \;\longrightarrow\; A\otimes_k B\ .$$

According to 6.1.b, we obtain an open epimorphism

$$\mathcal{C}^\infty(\mathbb{R}^{n+m}) \;=\; \mathcal{C}^\infty(\mathbb{R}^n)\,\widehat{\otimes}_{\mathbb{R}}\,\mathcal{C}^\infty(\mathbb{R}^m) \;\longrightarrow\; A\widehat{\otimes}_k B\ .$$

Its kernel is a closed ideal and we conclude that $A\widehat{\otimes}_k B$ is a differentiable algebra.

□

7 Fibred Products

In this chapter, we shall prove the existence of finite fibred products in the category of differentiable spaces. Let $\varphi_1\colon X_1 \to S$ and $\varphi_2\colon X_2 \to S$ be morphisms of differentiable spaces. A differentiable structure is given on the topological fibred product $X_1 \times_S X_2 := \{(x_1, x_2) \in X_1 \times X_2 \colon\ \varphi_1(x_1) = \varphi_2(x_2)\}$ satisfying the desired universal property:

$$\mathrm{Hom}\,(T, X_1 \times_S X_2) \;=\; \mathrm{Hom}\,(T, X_1) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}\,(T, X_2)$$

for any differentiable space T.

Let us summarize the construction. If $W \subseteq S$, $U_1 \subseteq \varphi_1^{-1}W$ and $U_2 \subseteq \varphi_2^{-1}W$ are affine open subsets, then the morphisms $\varphi_1\colon U_1 \to W$ and $\varphi_2\colon U_2 \to W$ are defined by certain morphisms of differentiable algebras $k \to A_1$ and $k \to A_2$. Then the differentiable algebra $A_1 \widehat{\otimes}_k A_2$ defines a differentiable structure on the open subset $U_1 \times_W U_2 \subseteq X_1 \times_S X_2$, since we have

$$U_1 \times_W U_2 \;=\; \mathrm{Spec}_r\,(A_1 \widehat{\otimes}_k A_2)\ .$$

This type of affine open subsets defines by "recollement" the differentiable structure on $X_1 \times_S X_2$.

7.1 Functor of Points of a Differentiable Space

Definition. Let X be a differentiable space. **Points of X parametrized** by a differentiable space T, or **T-points**, are defined to be morphisms $x\colon T \to X$. The set of all T-points of X is denoted by

$$X^\bullet(T) \;:=\; \mathrm{Hom}\,(T, X)\ .$$

Any morphism $\psi\colon T' \to T$ induces a natural map

$$\psi^*\colon X^\bullet(T) \longrightarrow X^\bullet(T') \quad , \quad \psi^*(x) := x \circ \psi\ ,$$

and the T'-point $\psi^*(x)$, also denoted by $x|_{T'}$ or simply x if no confusion is possible, is said to be the **specialization** of x at T'. So we obtain a contravariant functor $X^\bullet$ on the category **DifSp** of differentiable spaces, with values in the category of sets. This functor is said to be the **functor of points of X**.

J.A. Navarro González and J.B. Sancho de Salas: LNM 1824, pp. 79–87, 2003.

Let $p = \operatorname{Spec}_r \mathbb{R}$ be the one-point space. Then $X^\bullet(p)$ is the set of ordinary points of X.

The identity map $\tilde{x}\colon X \to X$ is said to be the **generic point** of X, since any other parametrized point $q\colon T \to X$ is a specialization of the generic point:

$$\tilde{x}|_T \;=\; q^*(\tilde{x}) \;=\; (Id)\circ q \;=\; q.$$

Definition. A contravariant functor F on the category **DifSp** of differentiable spaces, with values in the category of sets, is said to be a **sheaf** of sets on **DifSp** if, for any differentiable space T, the functor $U \rightsquigarrow F(U)$ is a sheaf of sets on T. That is to say, if $\{U_i\}$ is an open cover of T, then the following sequence is exact (the first map is injective and its image is just the subset where the two other maps coincide):

$$F(T) \;\to\; \prod_i F(U_i) \;\rightrightarrows\; \prod_{i,j} F(U_i \cap U_j)\ ,$$

where the two maps from $\prod_i F(U_i)$ to $\prod_{i,j} F(U_i \cap U_j)$ are induced by the maps $F(U_i) \to F(U_i \cap U_j)$ and $F(U_j) \to F(U_i \cap U_j)$ corresponding to the inclusion morphisms $U_i \cap U_j \hookrightarrow U_i$ and $U_i \cap U_j \hookrightarrow U_j$ respectively.

The functor of points $X^\bullet$ of any differentiable space X is a sheaf on the category **DifSp**, because morphisms of differentiable spaces admit “recollement” (3.17).

On the other hand, any morphism $\varphi\colon X \to Y$ transforms parametrized points of X into parametrized points of Y:

$$\varphi\colon X^\bullet(T) \longrightarrow Y^\bullet(T) \quad , \quad \varphi(p) := \varphi\circ p\ ,$$

so that φ defines a morphism of functors $\varphi\colon X^\bullet \to Y^\bullet$. It is a crucial fact that so we may understand the category of differentiable spaces **DifSp** as a full subcategory of the category of sheaves of sets on **DifSp**; that is to say, each differentiable space X is fully determined by its functor of points $X^\bullet$:

Theorem 7.1 (Yoneda's lemma). *Let F be a contravariant functor from the category of differentiable spaces to the category of sets and let X be a differentiable space. Any element $\xi \in F(X)$ defines a morphism of functors $\xi\colon X^\bullet \to F$, $\xi(p) := (Fp)(\xi)$, and each morphism of functors $X^\bullet \to F$ is defined by a unique element $\xi \in F(X)$, i.e.,*

$$F(X) \;=\; \operatorname{Hom}(X^\bullet, F)\ .$$

In particular, any morphism of functors $X^\bullet \to Y^\bullet$ (i.e., any morphism of sheaves) comes from a unique morphism of differentiable spaces $X \to Y$:

$$\operatorname{Hom}(X, Y) \;=\; \operatorname{Hom}(X^\bullet, Y^\bullet)\ .$$

Proof. The uniqueness is obvious, since we have $\xi(\tilde{x}) = \xi$ for the generic point $\tilde{x}$ of X.

Let us prove the existence. Given a morphism of functors $t\colon X^\bullet \to F$, we consider the generic point $\tilde{x} \in X^\bullet(X)$ and its image $\xi := t(\tilde{x}) \in F(X)$. Now, for any other point $p\colon T \to X$ we have a commutative square (because t is a morphism of functors)

$$\begin{array}{ccc} X^\bullet(X) & \xrightarrow{p^*} & X^\bullet(T) \\ \downarrow t & & \downarrow t \\ F(X) & \xrightarrow{Fp} & F(T) \end{array}$$

and we conclude that

$$t(p) \;=\; t(p^*(\tilde{x})) \;=\; (Fp)(t(\tilde{x})) \;=\; (Fp)(\xi) \;=\; \xi(p)\;.$$

That is to say, t is just the morphism of functors induced by ξ.

□

This important result (valid in arbitrary categories) shows that, in order to determine a differentiable space X, it is enough to give its (parametrized) points, and to determine a morphism of differentiable spaces $X \to Y$ we only have to known its action on the (parametrized) points of X.

Definition. Let F be a contravariant functor from the category of differentiable spaces to the category of sets. F is said to be **representable** by a differentiable space X if there exists an isomorphism of functors $X^\bullet \simeq F$ (defined by some element $\xi \in F(X)$ according to 7.1).

A subfunctor $F_i \subseteq F$ is said to be **open** when, for any differentiable space T and any morphism of functors $T^\bullet \to F$, the subfunctor $F_i \times_F T^\bullet \subseteq T^\bullet$ coincides with the subfunctor $T_i^\bullet \subseteq T^\bullet$ corresponding to some open differentiable subspace $T_i \subseteq T$. A family $\{F_i\}$ of open subfunctors of F is said to be an **open cover** of F if $\{T_i\}$ is an open cover of T for any differentiable space T.

Theorem 7.2. *Let F be a sheaf on the category **DifSp** of differentiable spaces with values in the category of sets. If there is an open cover $\{F_i\}$ of F such that F_i is representable for any index i, then F is representable.*

Proof. By hypothesis $F_i = X_i^\bullet$ for some differentiable space X_i. Let $p = \mathrm{Spec}_r\,\mathbb{R}$ be the one-point space. We may consider each X_i as a subset of $F(p)$ since $X_i = X_i^\bullet(p) = F_i(p) \subseteq F(p)$. Since F_i is an open subfunctor of F, we have

$$X_i^\bullet \cap X_j^\bullet \;=\; F_i \cap F_j \;=\; F_i \times_F F_j \;=\; F_i \times_F X_j^\bullet \;=\; X_{ij}^\bullet$$

for some open differentiable subspace $X_{ij} \subseteq X_j$; in particular, $X_i \cap X_j = X_{ij}$ as subsets of $F(p)$. Moreover, $X_i^\bullet \cap X_j^\bullet = X_j^\bullet \cap X_i^\bullet$, hence $X_{ij} = X_{ji}$ (isomorphism of differentiable spaces) and we obtain by "recollement" a differentiable space $X := \bigcup_i X_i$.

Now, we have to show that $F = X^\bullet$. Let T be a differentiable space. A morphism $T^\bullet \to F$ induces morphisms $T_i^\bullet = F_i \times_F T^\bullet \to F_i = X_i^\bullet \hookrightarrow X^\bullet$ and, since $X^\bullet$ is a sheaf, we obtain a morphism $T^\bullet \to X^\bullet$. Conversely, a morphism

$T^\bullet \to X^\bullet$ induces morphisms $T_i^\bullet = T^\bullet \times_{X^\bullet} X_i^\bullet \to X_i^\bullet = F_i \hookrightarrow F$ and, since F is a sheaf, we obtain a morphism $T^\bullet \to F$. In conclusion,

$$X^\bullet(T) \;=\; \mathrm{Hom}(T^\bullet, X^\bullet) \;=\; \mathrm{Hom}(T^\bullet, F) \;=\; F(T)\,.$$

□

7.2 Fibred Products

Definition. Let $\phi_1\colon X_1 \to S$, $\phi_2\colon X_2 \to S$ be morphisms of differentiable spaces. A differentiable space $X_1 \times_S X_2$, endowed with morphisms $p_1\colon X_1 \times_S X_2 \to X_1$, $p_2\colon X_1 \times_S X_2 \to X_2$ is said to be the **fibred product** of X_1 and X_2 over S if it satisfies the following universal property: $\phi_1 p_1 = \phi_2 p_2$ and there exists a bijection

$$\begin{array}{ccc} \mathrm{Hom}\,(T, X_1 \times_S X_2) & = & \mathrm{Hom}\,(T, X_1) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}\,(T, X_2) \\ \varphi & \mapsto & (\, p_1\varphi\,,\, p_2\varphi\,) \end{array}$$

for any differentiable space T.

The **direct product** of X_1 and X_2 is defined to be the fibred product over the one-point space $p = \mathrm{Spec}_r\,\mathbb{R}$, and we denote it by $X_1 \times X_2 := X_1 \times_p X_2$.

In other words, the fibred product $X_1 \times_S X_2$ represents the functor $X_1^\bullet \times_{S^\bullet} X_2^\bullet$, that is to say,

$$(X_1 \times_S X_2)^\bullet \;=\; X_1^\bullet \times_{S^\bullet} X_2^\bullet\,.$$

In particular, the direct product $X_1 \times X_2$ represents the functor $X_1^\bullet \times X_2^\bullet$.

Theorem 7.3. *Let $X_1 \to S$ and $X_2 \to S$ be morphisms of affine differentiable spaces, defined by some morphisms of differentiable algebras $k \to A_1$, $k \to A_2$. Then $X_1 \times_S X_2$ is the affine differentiable space defined by the differentiable algebra $A_1 \widehat{\otimes}_k A_2$:*

$$(\mathrm{Spec}_r\, A_1) \times_{\mathrm{Spec}_r k} (\mathrm{Spec}_r\, A_2) \;=\; \mathrm{Spec}_r (A_1 \widehat{\otimes}_k A_2)\,.$$

Proof. $A_1 \widehat{\otimes}_k A_2$ is a differentiable algebra by 6.13, hence $Z := \mathrm{Spec}_r\,(A_1 \widehat{\otimes}_k A_2)$ is an affine differentiable space. The natural morphisms $A_1 \to A_1 \widehat{\otimes}_k A_2$ and $A_2 \to A_1 \widehat{\otimes}_k A_2$ induce morphisms $p_1\colon Z \to X_1$ and $p_2\colon Z \to X_2$; hence we have a morphism of sheaves

$$Z^\bullet \;\longrightarrow\; X_1^\bullet \times_{S^\bullet} X_2^\bullet \quad , \quad z \mapsto (p_1(z), p_2(z))\,.$$

We have to prove that it is an isomorphism. Since the question is local, it is sufficient to see that the map

$$Z^\bullet(T) \;\longrightarrow\; X_1^\bullet(T) \times_{S^\bullet(T)} X_2^\bullet(T)$$

is a bijection for any affine differentiable space $T = \operatorname{Spec}_r C$. But this is clear since $A_1 \widehat{\otimes}_k A_2$ is the coproduct of the morphisms $k \to A_1$ and $k \to A_2$ in the category of complete locally m-convex algebras:

$$\operatorname{Hom}_{m\text{-alg}}(A_1 \widehat{\otimes}_k A_2, C) \;=\; \operatorname{Hom}_{m\text{-alg}}(A_1, C) \times_{\operatorname{Hom}_{m\text{-alg}}(k,C)} \operatorname{Hom}_{m\text{-alg}}(A_2, C) \;.$$

$\square$

Remark 7.4. We have $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ (direct product in the category of differentiable spaces).

Proof. It is a direct consequence of 3.19. Another proof: Combine 7.3 and 6.12. $\square$

Remark 7.5. With the notations of 7.3, the topological space underlying the affine differentiable space $X_1 \times_S X_2$ is the fibred product of the underlying topological spaces.

Proof. Let $A_1 = \mathcal{C}^\infty(\mathbb{R}^{n_1})/\mathfrak{a}_1$ and $A_2 = \mathcal{C}^\infty(\mathbb{R}^{n_2})/\mathfrak{a}_2$, hence we have closed embeddings $X_1 = (\mathfrak{a}_1)_0 \subseteq \mathbb{R}^{n_1}$, $X_2 = (\mathfrak{a}_2)_0 \subseteq \mathbb{R}^{n_2}$. Let $p = \operatorname{Spec}_r \mathbb{R}$ be the one-point space. The equality

$$(X_1 \times_S X_2)^\bullet(p) \;=\; X_1^\bullet(p) \times_{S^\bullet(p)} X_2^\bullet(p)$$

means that the underlying set of $X_1 \times_S X_2$ coincides with the fibred product of the underlying sets. To conclude it suffices to show that the natural morphism

$$X_1 \times_S X_2 \hookrightarrow X_1 \times X_2 \hookrightarrow \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \;=\; \mathbb{R}^{n_1+n_2}$$

is a closed embedding. By 6.1.b, $A_1 \widehat{\otimes}_{\mathbb{R}} A_2 \to A_1 \widehat{\otimes}_k A_2$ is an epimorphism, hence $X_1 \times_S X_2 \hookrightarrow X_1 \times X_2$ is a closed embedding. Finally, we have that $\mathcal{C}^\infty(\mathbb{R}^{n_1}) \widehat{\otimes}_{\mathbb{R}} \mathcal{C}^\infty(\mathbb{R}^{n_2}) \to A_1 \widehat{\otimes}_{\mathbb{R}} A_2$ is an epimorphism (6.3.b), and we conclude that $X_1 \times X_2 \hookrightarrow \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^{n_1+n_2}$ also is a closed embedding.

$\square$

Theorem 7.6. *The category of differentiable spaces has finite fibred products.*

Proof. Let $\phi_1 \colon X \to S$, $\phi_2 \colon Y \to S$ be morphisms of differentiable spaces. We apply 7.2 to show that the functor $F = X^\bullet \times_{S^\bullet} Y^\bullet$ is representable. Considering affine open covers

$$S = \bigcup_k W_k \quad , \quad \phi_1^{-1} W_k = \bigcup_i U_{ik} \quad , \quad \phi_2^{-1} W_k = \bigcup_j V_{jk} \;,$$

we obtain an open cover of F by the subfunctors $U_{ik}^\bullet \times_{W_k^\bullet} V_{jk}^\bullet \subseteq F$, which are representable by 7.3. Finally, F is a sheaf since it is a fibred product of sheaves. $\square$

Remark 7.7. The topological space underlying a finite fibred product of differentiable spaces is the fibred product of the underlying topological spaces.

Proof. With the notations of the proof of 7.6, we have $X \times_S Y = \bigcup (U_{ik} \times_{W_k} V_{jk})$, so that we have to consider only the affine case. We conclude by 7.5.

$\square$

7.3 Fibred Products of Embeddings

Lemma 7.8. *Let $\phi_1\colon X \to S$, $\phi_2\colon Y \to S$ be morphisms of differentiable spaces, let $(x,y) \in X \times_S Y$ and let $s = \phi_1(x) = \phi_2(y)$. The linear map $p_{1*} \times p_{2*}$ defines an isomorphism*

$$T_{(x,y)}(X \times_S Y) \;=\; (T_x X) \times_{T_s S} (T_y Y)\ .$$

Proof. Let $\mathbf{v}_p := \operatorname{Spec}_r \mathbb{R}[\varepsilon]$, where $\varepsilon^2 = 0$. To prove that $p_{1*} \times p_{2*}$ is bijective, note that the bijection

$$(X \times_S Y)^\bullet(\mathbf{v}_p) \;=\; X^\bullet(\mathbf{v}_p) \times_{S^\bullet(\mathbf{v}_p)} (Y)^\bullet(\mathbf{v}_p)$$

takes points infinitely near to (x,y) into pairs (u,v) where u is infinitely near to x and v is infinitely near to y. We conclude by 5.16.

□

Let $\phi'\colon X' \to S$, $\psi'\colon Y' \to S$, $\phi\colon X \to X'$ and $\psi\colon Y \to Y'$ be morphisms of differentiable spaces. By 7.1, the morphism of functors

$$X^\bullet \times_{S^\bullet} Y^\bullet \longrightarrow X'^\bullet \times_{S^\bullet} Y'^\bullet \quad , \quad (x,y) \mapsto (\phi(x), \psi(y)),$$

defines a morphism $\phi \times_S \psi\colon X \times_S Y \longrightarrow X' \times_S Y'$.

Proposition 7.9. *If $\phi\colon X \hookrightarrow X'$ and $\psi\colon Y \hookrightarrow Y'$ are embeddings, then so is $\phi \times_S \psi\colon X \times_S Y \hookrightarrow X' \times_S Y'$.*

Proof. With the notations of 7.8, the following square is commutative:

$$\begin{array}{ccc} (T_x X) \times_{T_s S} (T_y Y) & = & T_{(x,y)}(X \times_S Y) \\ \Big\downarrow {\scriptstyle \phi_* \times \psi_*} & & \Big\downarrow {\scriptstyle (\phi\times\psi)_*} \\ (T_x X') \times_{T_s S} (T_y Y') & = & T_{(x,y)}(X' \times_S Y') \end{array}$$

By hypothesis, ϕ_* and ψ_* are injective, hence so is $\phi_* \times \psi_*$. Applying 5.24 we conclude that $\phi \times_S \psi$ is an embedding, because

$$\phi \times_S \psi\colon X \times_S Y \longrightarrow (\phi \times_S \psi)(X \times_S Y) \;=\; \phi(X) \times_S \psi(Y)$$

is a homeomorphism, since so are $\phi\colon X \to \phi(X)$ and $\psi\colon Y \to \psi(Y)$.

□

Definition. The above result let us define the intersection of differentiable subspaces. Let $i\colon Y \hookrightarrow X$ and $j\colon Z \hookrightarrow X$ be differentiable subspaces of a differentiable space X. According to 7.9, the morphism $i \times_X j\colon Y \times_X Z \hookrightarrow X \times_X X = X$ is an embedding; hence it defines an isomorphism of $Y \times_X Z$ onto a differentiable subspace of X, that we call the **intersection** of Y and Z in X, and we denote by $Y \cap Z$ since $(Y \cap Z)^\bullet = Y^\bullet \cap Z^\bullet$.

Corollary 7.10. *If* $\phi\colon X \hookrightarrow X'$ *and* $\psi\colon Y \hookrightarrow Y'$ *are closed embeddings, then so is* $\phi \times_S \psi\colon X \times_S Y \hookrightarrow X' \times_S Y'$.

Proof. $(\phi \times_S \psi)(X \times_S Y) = \phi(X) \times_S \psi(Y)$ is closed in $X' \times_S Y'$ when $\phi(X)$ is closed in X' and $\psi(Y)$ is closed in Y'.

□

Corollary 7.11. *If* $\phi\colon U \hookrightarrow X$ *and* $\psi\colon V \hookrightarrow Y$ *are open embeddings, then so is* $\phi \times_S \psi\colon U \times_S V \hookrightarrow X \times_S Y$.

Proof. It is enough to show that $(U \times_S V, \mathcal{O}_{X \times_S Y}|_{U \times_S V})$ has the universal property of the fibred product $U \times_S V$, and this is a direct consequence of the universal property of open subspaces.

□

Lemma 7.12. $U^r_{(x,y)}(X \times_S Y) \subseteq U^r_x(X) \times_S U^r_y(Y)$ *for any* $r \in \mathbb{N}$.

Proof. The canonical projections $p_1\colon X \times_S Y \to X$, $p_2\colon X \times_S Y \to Y$ induce morphisms

$$U^r_{(x,y)}(X \times_S Y) \longrightarrow U^r_x(X) \quad , \quad U^r_{(x,y)}(X \times_S Y) \longrightarrow U^r_y(Y) \, ,$$

which define a natural morphism $U^r_{(x,y)}(X \times_S Y) \to U^r_x(X) \times_S U^r_y(Y)$ compatible with the respective inclusion morphisms into $X \times_S Y$.

□

Lemma 7.13. *Let* X, Y *be differentiable spaces. If a differentiable function* f *on* $X \times Y$ *vanishes on* $X \times U^r_y$ *for any* $r \in \mathbb{N}$, $y \in Y$, *then* $f = 0$.

Proof. If $f = 0$ on $X \times U^r_y$, then f vanishes on $U^r_x \times U^r_y$ for any $x \in X$; hence $f = 0$ on $U^r_{(x,y)}$ by 7.12. Now 5.11 let us conclude that $f = 0$.

□

Lemma 7.14. *If* A *is a rational finite* $\mathbb{R}$*-algebra and* B *is a differentiable algebra, then* $A \otimes_{\mathbb{R}} B$ *is a differentiable algebra and*

$$(\mathrm{Spec}_r\, A) \times (\mathrm{Spec}_r\, B) \;=\; \mathrm{Spec}_r\, (A \otimes_{\mathbb{R}} B) \, .$$

Proof. By 9.2, A is a differentiable algebra. Since A is a finite dimensional vector space, it is clear that $A \otimes_{\mathbb{R}} B = (\mathbb{R} \oplus \cdots \oplus \mathbb{R}) \otimes_{\mathbb{R}} B = B \oplus \cdots \oplus B$ is complete, i.e., $A \otimes_{\mathbb{R}} B = A \widehat{\otimes}_{\mathbb{R}} B$; hence it is a differentiable algebra by 6.13. We conclude by 7.3.

□

Lemma 7.15. *Let* X, Y *be differentiable spaces and let* f *be a differentiable function on* $X \times Y$. *If* Y *is reduced and* f *vanishes on* $X \times y$ *for any* $y \in Y$, *then* $f = 0$.

Proof. Considering an affine open cover of Y, we may assume that $Y = \mathrm{Spec}_r\, B$.

Let us assume that $X = \mathrm{Spec}_r\, A$ for some rational finite $\mathbb{R}$-algebra A, and let us consider a basis $\{f_1, \ldots, f_n\}$ of A as a real vector space. By 7.14, in this case we have

$$\mathcal{O}_{X\times Y}(X \times Y) \;=\; A \otimes_{\mathbb{R}} B \;=\; Bf_1 \oplus \ldots \oplus Bf_n \;,$$

so that there exist differentiable functions $g_1, \ldots, g_n \in B$ on Y such that

$$f = f_1 g_1 + \ldots + f_n g_n \;.$$

If f vanishes on $X \times y$, then $g_1(y) = \ldots = g_n(y) = 0$. Therefore, $g_1, \ldots, g_n$ vanish at any point of Y and, Y being reduced, we have $g_1 = \ldots = g_n = 0$ and we conclude that $f = 0$.

In the general case, if f vanishes on $X \times y$, then f vanishes on $U_x^r \times y$ for any $r \in \mathbb{N}$, $x \in X$. Since $U_x^r = \mathrm{Spec}_r\,(\mathcal{O}_{X,x}/\mathfrak{m}_x^{r+1})$ and $\mathcal{O}_{X,x}/\mathfrak{m}_x^{r+1}$ is a rational finite $\mathbb{R}$-algebra, the former case let us obtain that f vanishes on $U_x^r \times Y$ for any $r \in \mathbb{N}$, $x \in X$. Now 7.13 let us conclude that $f = 0$.

□

Proposition 7.16. *Let X, Y, T be differentiable spaces and let $\varphi, \phi\colon X \times T \to Y$ be morphisms. If T is reduced and $\varphi|_{X\times t} = \phi|_{X\times t}$ for any $t \in T$, then $\varphi = \phi$.*

Proof. Condition $\varphi|_{X\times t} = \phi|_{X\times t}$ implies that φ and ϕ define the same continuous map $X \times T \to Y$. Now, considering an affine open cover of Y, we may assume that $Y = \mathrm{Spec}_r\, A$. In such a case, by 3.18, we only have to prove that $\varphi^* f = \phi^* f$ for any differentiable function $f \in A$. By hypothesis $(\varphi^* f)|_{X\times t} = (\phi^* f)|_{X\times t}$ for any $t \in T$ and 7.15 let us conclude that $\varphi^* f = \phi^* f$.

□

7.4 Base Change of Differentiable Spaces

Definition. Let S be a differentiable space. A **differentiable space over** S, or just a **differentiable** S**-space**, is defined to be any morphism of differentiable spaces $\phi\colon X \to S$. If $\phi_1\colon X \to S$ and $\phi_2\colon Y \to S$ are differentiable S-spaces, a morphism of differentiable spaces $\varphi\colon X \to Y$ is said to be a S**-morphism** when $\phi_1 = \phi_2 \circ \varphi$. The set of all S-morphisms $X \to Y$ will be denoted by $\mathrm{Hom}_S(X, Y)$.

Differentiable S-spaces, with S-morphisms, constitute a category $\mathbf{DifSp}_{/S}$, and the universal property of the fibred product $X \times_S Y$ shows that it is the direct product of X and Y in such category. Each differentiable space X admits a unique morphism $X \to p = \mathrm{Spec}_r\, \mathbb{R}$, so that the category $\mathbf{DifSp}_{/p}$ of differentiable spaces over the one-point space p is canonically isomorphic to the category **DifSp** of differentiable spaces.

Definition. Let X be a differentiable S-space and let $\phi\colon S' \to S$ be a morphism of differentiable spaces. The second projection $p_2\colon X \times_S S' \to S'$ defines a structure of S'-space on $X_{S'} := X \times_S S'$, and we say that the S'-space $X_{S'}$ is obtained from the S-space X by the **base change** $\phi\colon S' \to S$.

If $\varphi\colon X \to Y$ is a S-morphism, then $\varphi_{S'} := \varphi \times_S Id\colon X \times_S S' \to Y \times_S S'$ is a S'-morphism, and we say that we obtain it from φ by the base change $\phi\colon S' \to S$.

The universal property of the fibred product shows that base changes are transitive:

$$(X_{S'})_{S''} \;=\; X_{S''}$$

(for any morphism $S'' \to S'$) and commute with fibred products:

$$(X \times_S Y)_{S'} \;=\; (X_{S'}) \times_{S'} (Y_{S'})\ .$$

On the other hand, according to 7.9, 7.10 and 7.11, the concepts of embedding, closed embedding and open embedding are stable under base changes. In particular, if Y is a differentiable subspace of X, then $Y_{S'}$ may be canonically identified with a differentiable subspace of $X_{S'}$.

Definition. Let $i\colon Y \hookrightarrow X$ be a differentiable subspace. If $\varphi\colon Z \to X$ is a morphism of differentiable spaces, then $Id \times_X i\colon Z \times_X Y \hookrightarrow Z \times_X X = Z$ is an embedding by 7.9. It follows that we may identify $Z \times_X Y$ with a differentiable subspace of Z, called the **inverse image** of Y by φ, and we denote it by $\varphi^{-1}(Y)$.

According to 7.10 and 7.11, if Y is a closed (resp. open) differentiable subspace of X, then $\varphi^{-1}(Y)$ is a closed (resp. open) differentiable subspace of Z.

In the particular case of a point $x \in X$, we say that $\varphi^{-1}(x)$ is the **fibre** of φ over x, and it is a closed differentiable subspace of Z.

The equality $\varphi^{-1}(Y) = Z \times_X Y$ states that (parametrized) points of the inverse image $\varphi^{-1}(Y)$ are just points of Z whose image by φ is a point of Y, i.e.,

$$(\varphi^{-1}Y)^\bullet(T) = \mathrm{Hom}(T, Z \times_X Y) = \{z \in Z^\bullet(T) : \varphi(z) \in Y^\bullet(T)\}$$

for any differentiable space T.

8 Topological Localization

Let A be a differentiable algebra. The Localization theorem for differentiable algebras (chapter 3) states that the presheaf $U \rightsquigarrow A_U$ is a sheaf on $\operatorname{Spec}_r A$, i.e., $\widetilde{A}(U) = A_U$. In this chapter we extend the Localization theorem to Fréchet modules [35, 41]: If M is a Fréchet A-module, then the presheaf $U \rightsquigarrow M_U$ is a sheaf on $\operatorname{Spec}_r A$, i.e.,

$$\widetilde{M}(U) \;=\; M_U \,.$$

The localization of any locally m-convex A-module (with respect to a multiplicative system) inherits a natural locally m-convex topology, called the localization topology [52]. We shall show that if M is a Fréchet A-module, then M_U is a Fréchet A_U-module for any open subset U of $\operatorname{Spec}_r A$. In the particular case $M = A$, the localization topology in A_U coincides with the canonical Fréchet topology of A_U as a differentiable algebra.

8.1 Localization Topology

If S is a multiplicative set of a ring A, then the localization of A by S (or ring of fractions, see [1] chapter 3) will be denoted by $A_S = S^{-1}A$, and we say that the canonical morphism $\gamma\colon A \to A_S$, $\gamma(a) := a/1$, is the **localization morphism**. All the elements of S are invertible in A_S; that is to say, $\gamma(S) \subseteq (A_S)^*$, and any other morphism of rings $f\colon A \to B$ such that $f(S) \subseteq B^*$ factors, in a unique way, through γ.

If M is an A-module, then $M_S = S^{-1}M$ will denote the localization of M by S ([1] chapter 3), and the canonical morphism $\gamma'\colon M \to M_S$, $\gamma'(m) := m/1$, is said to be the **localization morphism**. M_S is an A_S-module and any morphism of A-modules $h\colon M \to N$ into an A_S-module N factors, in a unique way, through γ'.

Let A be a locally m-convex algebra. If $j\colon A \to B$ is a morphism of $\mathbb{R}$-algebras, then there exists on B the finer locally m-convex topology such that j is continuous: it is defined by all the submultiplicative seminorms q such that $j\colon A \to (B,q)$ is continuous. With this topology, a morphism of $\mathbb{R}$-algebras $f\colon B \to C$ into a locally m-convex algebra C is continuous if and only if so is $fj\colon A \to C$.

Definition. Let S be a multiplicative system of a locally m-convex algebra A. The **localization topology** in A_S is defined to be the finer locally m-convex

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topology such that the localization morphism $\gamma\colon A \to A_S$ is continuous, so that the locally m-convex algebra A_S is characterized by the following universal property:

Let $j\colon A \to B$ be a morphism of locally m-convex algebras. If $j(s)$ is invertible in B for any $s \in S$, then there exists a unique morphism of locally m-convex algebras $j'\colon A_S \to B$ such that $j'(a/1) = j(a)$ for any $a \in A$:

$$\mathrm{Hom}_{m\text{-alg}}(A_S, B) \;=\; \{j \in \mathrm{Hom}_{m\text{-alg}}(A, B) :\; j(S) \subseteq B^*\}\;.$$

In general, if T is a multiplicative system of B and $j(S) \subseteq T$, this universal property shows that the morphism $j'\colon A_S \to B_T$, $j'(a/s) = j(a)/j(s)$, is continuous.

Definition. Let M be a locally convex A-module. The **localization topology** in M_S is defined to be the finer topology of locally convex A_S-module such that the localization morphism $\gamma'\colon M \to M_S$ is continuous, so that the locally convex A_S-module M_S is characterized by the following universal property:

Let N be a locally convex A_S-module. If $h\colon M \to N$ is a continuous morphism of A-modules, then there exists a unique continuous morphism of A_S-modules $h'\colon M_S \to N$ such that $h'(m/1) = h(m)$ for any $m \in M$:

$$\mathrm{Hom}_{A_S}(M_S, N) \;=\; \mathrm{Hom}_A(M, N)\;.$$

If $h\colon M \to N$ is a morphism of locally convex A-modules, then the morphism of A_S-modules $h_S\colon M_S \to N_S$, $h_S(m/s) = h(m)/s$, is continuous by the universal property of M_S.

Proposition 8.1. *Let S be a multiplicative set of a locally m-convex algebra A. If M is a locally convex A-module, then we have a topological isomorphism:*

$$M_S \;=\; M \otimes_A A_S\;.$$

Proof. For any locally convex A_S-module N we have

$$\mathrm{Hom}_{A_S}(M_S, N) \;=\; \mathrm{Hom}_A(M, N) \;=\; \mathrm{Hom}_{A_S}(M \otimes_A A_S, N)\;.$$

□

Proposition 8.2. *Let $S \subseteq T$ be multiplicative sets of a locally m-convex algebra A. If M is a locally convex A-module, then we have a topological isomorphism*

$$(M_S)_T \;=\; M_T\;.$$

In particular, if the natural morphism $A_S \to A_T$ is an algebraic isomorphism, then the natural isomorphism $M_S \to M_T$ also is a homeomorphism.

Proof. It is clear that the algebraic isomorphism $(A_S)_T = A_T$ is a homeomorphism. Moreover, the algebraic isomorphism $(M_S)_T = M_T$ preserves the localization morphisms $M \to M_S \to (M_S)_T$ and $M \to M_T$; hence it is a homeomorphism, since both A_T-modules are endowed with the finer topology of locally convex A_T-module such that these morphisms are continuous.

□

Proposition 8.3. *Let S be a multiplicative set of a locally m-convex algebra A and let M be a locally convex A-module. For any submodule N of M we have a topological isomorphism*

$$(M/N)_S \;=\; M_S/N_S \;.$$

Proof. The natural morphism $M/N \to M_S/N_S$ is continuous and M_S/N_S is a locally convex A_S-module; hence the natural morphism $(M/N)_S \to M_S/N_S$ is continuous. On the other hand, the natural morphism $M_S \to (M/N)_S$ is continuous and it vanishes on N_S, so that it induces a continuous morphism $M_S/N_S \to (M/N)_S$.

□

Proposition 8.4. *Let $j\colon A \to B$ be a morphism of locally m-convex algebras and let S be a multiplicative set of A. The localization topology of the A-module B by S coincides with the localization topology of the algebra B by the multiplicative set $j(S)$:*

$$B_S \;=\; B_{j(S)}$$

and for any locally convex B-module N we have a topological isomorphism

$$N_S \;=\; N_{j(S)} \;.$$

Proof. The algebraic isomorphism $B \otimes_A A_S \to B_{j(S)}$ is continuous since so are $B \to B_{j(S)}$ and $A_S \to B_{j(S)}$. Moreover, $B \otimes_A A_S$ is a locally m-convex algebra and the natural morphism $B \to B \otimes_A A_S$, which is continuous, transforms any element of $j(S)$ into an invertible element, so that it induces a continuous morphism $B_{j(S)} \to B \otimes_A A_S$, which is just the inverse morphism.

Finally, we have

$$N_S \;=\; N \otimes_A A_S \;=\; N \otimes_B (B \otimes_A A_S) \;=\; N \otimes_B B_S \;=\; N \otimes_B B_{j(S)} = N_{j(S)} \;.$$

□

Theorem 8.5 ([52]). *Let S be a multiplicative set of a locally m-convex algebra A. If τ is a locally m-convex topology on A_S such that $\pi\colon A \times S \to (A_S, \tau)$, $\pi(a,s) = a/s$, is continuous and it admits a continuous section σ, then the localization topology coincides with τ, which is the final topology induced by π.*

If M is a locally convex A-module and τ' is a locally convex topology on M_S such that $\pi'\colon M \times S \to (M_S, \tau')$, $\pi'(m,s) = m/s$, is continuous and it admits a continuous section σ', then the localization topology coincides with τ', which is the final topology induced by π'.

Proof. Since $\gamma\colon A \to (A_S, \tau)$ is continuous, the localization topology is finer than τ. Since $\pi\colon A \times S \to A_S$ is a continuous map, because it is the composition

$$A \times S \xrightarrow{\gamma\times\gamma} A_S \times (A_S)^* \xrightarrow{Id\times inv} A_S \times (A_S)^* \xrightarrow{\cdot} A_S \ ,$$

the final topology induced by π is finer than the localization topology. Finally, since $\pi\colon A \times S \to (A_S, \tau)$ admits a continuous section, τ is finer than the final topology.

Moreover, $\pi'\colon M \times S \to M_S$ is a continuous map, since it is the composition

$$M \times S \xrightarrow{\gamma'\times\gamma} M_S \times (A_S)^* \xrightarrow{Id\times inv} M_S \times (A_S)^* \xrightarrow{\cdot} M_S \ ,$$

so that the final topology induced by π' is finer than the localization topology. Since $\pi'\colon M \times S \to (M_S, \tau')$ admits a continuous section, τ' is finer than the final topology. If we prove that (M_S, τ') is a locally convex A_S-module, then the localization topology is finer than τ' and we conclude that these three topologies coincide. Now, it is enough to consider the following commutative diagram:

$$\begin{array}{ccc} A_S \times (M_S, \tau') & \xrightarrow{\cdot} & (M_S, \tau') \\ \Big\downarrow{\scriptstyle \sigma\times\sigma'} & & \Big\uparrow{\scriptstyle \pi'} \\ A \times S \times M \times S & \xrightarrow{P} & M \times S \end{array}$$

where $P(a, s, m, t) = (am, st)$ is clearly a continuous map.

□

8.2 Topological Localization of Differentiable Algebras

Lemma 8.6. *Let X a topological space, let F be a Fréchet vector space, and let $q_1 < q_2 < \ldots < q_r < \ldots$ be a fundamental system of seminorms of F. If $f_i\colon X \to F$ is a sequence of continuous maps such that $q_i(f_i(x)) \leq 2^{-i}$, then the following map $f\colon X \to F$ is continuous:*

$$f(x) = \sum_{i=1}^{\infty} f_i(x) \ .$$

Proof. Let us fix a point $a \in X$, a seminorm q_r and a real positive number ε. Let $m \geq r$ be a natural number such that $2^{-m} < \varepsilon/4$. The map $f_1 + \ldots + f_m\colon X \to F$ is continuous, so that there exists a neighbourhood U of a in X such that

$$q_r\left(\sum_{i=1}^{m} f_i(x) - \sum_{i=1}^{m} f_i(a)\right) < \frac{\varepsilon}{2}$$

for any $x \in U$. Hence

$$
\begin{aligned}
q_r\big(f(x)-f(a)\big) &= q_r\left(\sum_{i=1}^{\infty} f_i(x) - \sum_{i=1}^{\infty} f_i(a)\right)\\
&\le q_r\left(\sum_{i=1}^{m} f_n(x) - \sum_{i=1}^{m} f_n(a)\right) + q_r\left(\sum_{i>m} f_i(x) - \sum_{i>m} f_i(a)\right)\\
&< \frac{\varepsilon}{2} + \sum_{i=m+1}^{\infty}\frac{1}{2^i} + \sum_{i=m+1}^{\infty}\frac{1}{2^i} \;=\; \frac{\varepsilon}{2} + \frac{1}{2^m} + \frac{1}{2^m} < \varepsilon\,.
\end{aligned}
$$

□

Theorem 8.7 ([52]). *Let A be a differentiable algebra and let U be an open set in $X = \operatorname{Spec}_r A$. Let S be the multiplicative set of all elements $a \in A$ without zeros in U. The canonical topology of the differentiable algebra $A_U = A_S$ coincides with the localization topology and with the final topology induced by the map $A \times S \to A_U$, $(a,s) \mapsto a/s$.*

Proof. By 8.5, it is enough to prove that

$$\pi\colon A \times S \longrightarrow A_U \quad , \quad \pi(a,s) = a/s\,,$$

is a continuous map and that it admits a continuous section, when we consider on A_U the canonical topology.

The natural morphism $A \to A_U$ is continuous by 2.23. Then π is continuous because it is the following composition of continuous maps

$$A \times S \longrightarrow A_U \times (A_U)^* \xrightarrow{(Id,inv)} A_U \times (A_U)^* \stackrel{\cdot}{\to} A_U\ .$$

Let us show that π admits a continuous section. Let $\{K_n\}$ be a sequence of compact subsets of U such that $\bigcup_n K_n = U$ and $K_n \subseteq \overset{\circ}{K}_{n+1}$ for any index n. Let $b_n \in A$ such that $b_n \geq 0$ on X, $b_n(x) > 0$ for any $x \in K_n$ and $\operatorname{Supp} b_n \subseteq \overset{\circ}{K}_{n+1}$ (see 3.8). Note that for any $f \in A_U = \widetilde{A}(U)$ we have that $b_n f \in \widetilde{A}(X) = A$. Let $q_1 \leq q_2 \leq \dots$ be a fundamental system of seminorms of A. The map $A_U \to A \times S$, $f \mapsto (a(f), s(f))$, where

$$
\begin{aligned}
a(f) &= \sum_{i=1}^{\infty} 2^{-i}\frac{b_i f}{1+q_i(b_i f)+q_i(b_i)}\ ,\\
s(f) &= \sum_{i=1}^{\infty} 2^{-i}\frac{b_i}{1+q_i(b_i f)+q_i(b_i)}\ ,
\end{aligned}
$$

define a section of π since $f = a(f)/s(f)$. In fact, the above equalities remain valid in A_U (because $A \to A_U$ is continuous) and it is clear that $a(f) = f \cdot s(f)$ in A_U.

Finally, this section is continuous because so is each summand in $a(f)$ and $s(f)$, and we may apply 8.6. In fact

$$
\begin{aligned}
q_i\left(2^{-i}\frac{b_i f}{1+q_i(b_i f)+q_i(b_i)}\right) &\le 2^{-i}\ ,\\
q_i\left(2^{-i}\frac{b_i}{1+q_i(b_i f)+q_i(b_i)}\right) &\le 2^{-i}\ .
\end{aligned}
$$

□

Corollary 8.8. *Let $X = \operatorname{Spec}_r A \to S = \operatorname{Spec}_r k$, $Y = \operatorname{Spec}_r B \to S$ be morphisms of affine differentiable spaces. If $U \subset X$ and $V \subset Y$ are open sets, then*

$$A_U \widehat{\otimes}_k B_V \ = \ (A \widehat{\otimes}_k B)_{U \times_S V} \ .$$

Proof. According to 7.3, we have

$$\begin{aligned} U \times_S V &= \operatorname{Spec}_r (A_U) \times_S \operatorname{Spec}_r (B_V) \ = \ \operatorname{Spec}_r (A_U \widehat{\otimes}_k B_V) \ , \\ X \times_S Y &= \operatorname{Spec}_r (A \widehat{\otimes}_k B) \end{aligned}$$

and we may conclude by 7.11.

□

8.3 Localization of Fréchet Modules

Localization theorem for Fréchet modules ([41]). *Let A be a differentiable algebra and let M be a Fréchet A-module. The presheaf $U \rightsquigarrow M_U$ is a sheaf on $X = \operatorname{Spec}_r A$:*

$$M_U \ = \tilde{M}\,(U) \ .$$

In particular, $M \ = \tilde{M}\,(X)$.

Proof. (1) *The natural morphism $M_U \to \tilde{M}\,(U)$ is injective:*

If the germ of $m/s \in M_U$ vanishes at certain point $x \in U$, then there exists a function $a \in A$ such that $a(x) \neq 0$ and $am = 0$, since $\tilde{M}_x = M_x$. If the germ of m/s at any point of U is 0, then we may choose a sequence $\{a_i\}$ in A such that $a_i m = 0$, $a_i \geq 0$ on X and the interior sets of the supports $\operatorname{Supp} a_i$ cover U. Let $q_1 \leq q_2 \leq \ldots$ be seminorms defining the topology of A. The convergent series

$$\bar{s} \ = \sum_{i=1}^{\infty} 2^{-i} \frac{a_i}{1 + q_i(a_i)}$$

defines an element in A, which does not vanish at any point of U, such that

$$\bar{s}m \ = \sum_{i=1}^{\infty} 2^{-i} \frac{a_i m}{1 + q_i(a_i)} \ = \sum_{i=1}^{\infty} 0 \ = \ 0 \ .$$

We conclude that $m/s = m\bar{s}/s\bar{s} = 0$ in M_U.

(2) *Let $m = \sum_i m_i$ be a convergent series in M and let $x \in X$. If there exists a neighbourhood of x which intersects only a finite number of members of the family $\{\operatorname{Supp}(m_i)\}$, then $m_x = \sum_i (m_i)_x$:*

Let $a \in A$ be an element, whose support intersects only a finite number of members of such family, such that $a(x) \neq 0$. By step 1, the series $\sum_i am_i$ only has a finite number of non-zero summands. Now:

$$a_x m_x = (am)_x = \Big(\sum_i am_i\Big)_x = \sum_i (am_i)_x = \sum_i a_x (m_i)_x = a_x \Big(\sum_i (m_i)_x\Big)$$

and, a_x being invertible in the local ring A_x, we conclude that $m_x = \sum_i (m_i)_x$.

(3) *The natural morphism* $M_U \to \widetilde{M}(U)$ *is surjective:*

Let $\tilde{m} \in \widetilde{M}(U)$. Since the morphism $M \to M_x = \widetilde{M}_x$ is surjective (1.6), for any point $x \in U$ there exists a function $a \in A$ such that $a \geq 0$ on X, $a(x) \neq 0$ and $a\tilde{m}$ is a global section which is in M. We may choose a sequence $\{a_i\}$ of such functions, so that $\{\operatorname{Supp} a_i\}$ is a locally finite family whose interior sets cover U. Let $p_1 \leq p_2 \leq \ldots$ be seminorms defining the Fréchet topology of M and let $q_1 \leq q_2 \leq \ldots$ be seminorms defining the topology of A. The convergent series

$$m = \sum_{i=1}^{\infty} 2^{-i} \frac{a_i \tilde{m}}{1 + p_i(a_i \tilde{m}) + q_i(a_i)} ,$$

$$s = \sum_{i=1}^{\infty} 2^{-i} \frac{a_i}{1 + p_i(a_i \tilde{m}) + q_i(a_i)} ,$$

define an element $m \in M$ and a function $s \in A$ which does not vanish at any point of U, and we have $\tilde{m} = m/s$ because both sections have the same germ at any point of U by step 2.

(4) Finally, $\widetilde{M}(X) = M_X = M \otimes_A A_X = M \otimes_A A = M$.

□

Theorem 8.9 ([52]). *Let A be a differentiable algebra and let U be an open set in $X = \operatorname{Spec}_r A$. If M is a Fréchet A-module, then M_U is a Fréchet A_U-module with the localization topology, which coincides with the final topology induced by the map $\pi' \colon M \times S \to M_U$, $\pi'(m,s) = m/s$ (where S denotes the multiplicative set of all differentiable functions $f \in A$ without zeros in U).*

Proof. By 8.5, we have to prove that $\pi' \colon M \times S \to M_U$ is continuous and admits a continuous section, for a certain Fréchet topology on M_U.

Let $\{K_i\}$ be a sequence of compact subsets of U such that $\bigcup_i K_i = U$ and $K_i \subseteq U_{i+1}$ for each index i, where U_{i+1} is the interior subset of K_{i+1}. Let M_i be the submodule of M of all elements with null germ at any point of K_i. We have $\overline{M}_i \subseteq M_{i-1}$. In fact, given $x \in K_{i-1}$, let us consider a function $a \in A$ such that $a(x) \neq 0$ and $\operatorname{Supp} a \subseteq K_i$. If $m \in M_i$, then $am = 0$ since it has null germ at any point of X. By continuity, $a\overline{M}_i = 0$ and, since $a(x) \neq 0$, we conclude that any element of $\overline{M}_i$ has null germ at x. That is to say, $\overline{M}_i \subseteq M_{i-1}$ and the natural morphism $M \to M_{U_{i-1}} = \widetilde{M}(U_{i-1})$ factors through $M/\overline{M}_i$.

On the other hand, let W_i be the closure of the ideal of all elements in A vanishing on a neighbourhood of K_i, so that A/W_i is a differentiable algebra with real spectrum K_i. By 2.15, the natural morphism $A \to A/W_i$ factors through $A_{U_{i+1}}$. Now, $M \to M/\overline{M}_i$ factors through $M_{U_{i+1}}$ because $M/\overline{M}_i$ is an A/W_i-module. So we get a projective system:

$$\dots \longrightarrow M/\overline{M}_{i+2} \longrightarrow M_{U_{i+1}} \longrightarrow M/\overline{M}_i \longrightarrow M_{U_{i-1}} \longrightarrow \dots ,$$
$$M_U = \widetilde{M}(U) = \varprojlim_i \widetilde{M}(U_i) = \varprojlim_i M/\overline{M}_i$$

and, since any module $M/\overline{M}_i$ is Fréchet, we obtain a Fréchet topology τ' on M_U. Now we have to prove that the map

$$\pi' \colon M \times S \longrightarrow (M_U, \tau') \quad , \quad \pi'(m,s) = m/s ,$$

is continuous and admits a continuous section. It is continuous since so are the maps

$$M \times S \longrightarrow (M/\overline{M}_i) \times (A/W_i)^* \xrightarrow{Id \times inv} (M/\overline{M}_i) \times (A/W_i)^* \stackrel{\cdot}{\longrightarrow} M/\overline{M}_i \ .$$

Let us show that π' admits a continuous section. We choose non-negative functions $a_i \in A$ such that $a_{i,x} = 1$ for any $x \in K_{i-1}$ and $\operatorname{Supp} a_i \subseteq K_i$, then we define $\sigma \colon M_U \to M \times S$, $\tilde{m} \mapsto (m,s)$, where

$$m = \sum_{i=1}^{\infty} 2^{-i} \frac{a_i \tilde{m}}{1 + p_i(a_i\tilde{m}) + q_i(a_i)} \in M ,$$
$$s = \sum_{i=1}^{\infty} 2^{-i} \frac{a_i}{1 + p_i(a_i\tilde{m}) + q_i(a_i)} \in S .$$

Using step (2) of the former proof, it is clear that $\tilde{m} = m/s$, hence σ is a section of π'. Now we prove that σ is continuous. Since $\operatorname{Supp} a_i \subseteq K_i$, the map $\widetilde{M}(U) = \varprojlim M/\overline{M}_i \to M$, $\tilde{m} \mapsto a_i\tilde{m}$, is just the composition of the canonical map $\varprojlim M/\overline{M}_i \to M/\overline{M}_{i+1}$ with the continuous map $M/\overline{M}_{i+1} \to M/M_i \to M$, $[\tilde{m}] \mapsto a_i\tilde{m}$, hence $\widetilde{M}(U) \to M$, $\tilde{m} \mapsto a_i\tilde{m}$ is continuous. By 8.6, the maps $M_U \to M$, $\tilde{m} \mapsto m$, and $M_U \to S$, $\tilde{m} \mapsto s$, are continuous.

□

Corollary 8.10. *Let A be a differentiable algebra and let U be an open set in* $\operatorname{Spec}_r A$. *If N is a closed submodule of a Fréchet A-module M, then N_U is a closed submodule of M_U.*

Proof. The kernel of the natural morphism $p \colon M_U \to (M/N)_U = M_U/N_U$ is a closed submodule of M_U, hence it is Fréchet. The continuous algebraic isomorphism $N_U \to \operatorname{Ker} p$ is a homeomorphism since both modules are Fréchet.

□

Corollary 8.11. *Let A be a differentiable algebra and let U be an open set in* $\operatorname{Spec}_r A$. *If M is a Fréchet A-module, then we have topological isomorphisms:*

$$M_U = M \otimes_A A_U = M \widehat{\otimes}_A A_U .$$

Proof. By 8.1, we have a topological isomorphism $M_U = M \otimes_A A_U$. Moreover, M_U is complete according to the Localization theorem for Fréchet modules, and we conclude that $M \otimes_A A_U = M \widehat{\otimes}_A A_U$.

□

Corollary 8.12. *Let* $\varphi \colon \mathrm{Spec}_r B \to \mathrm{Spec}_r A$ *be a morphism of affine differentiable spaces, let* U *be an open set in* $\mathrm{Spec}_r A$ *and let* $V := \varphi^{-1}(U)$*. If* M *is a Fréchet* B*-module, then the localization of* M *in* V *coincides with the localization of the* A*-module* M *in* U*:*

$$M_U \; = \; M_V \; .$$

Therefore, $\varphi_* \widetilde{M}$ *is the sheaf associated to the Fréchet* A*-module* M*.*

Proof. The sheaf $\varphi_*(\widetilde{B})$ is an $\widetilde{A}$-module. By 3.11, it is the sheaf associated to the A-module $\Gamma(\mathrm{Spec}_r A, \varphi_*(\widetilde{B})) = \Gamma(\mathrm{Spec}_r B, \widetilde{B}) = B$, which is a Fréchet A-module. According to the Localization theorem for Fréchet modules, the module of sections on U coincides with B_U:

$$B_U \; = \; \Gamma(U, \varphi_*(\widetilde{B})) \; = \; \Gamma(\varphi^{-1}U, \widetilde{B}) \; = \widetilde{B}\,(V) \; = \; B_V \; .$$

Now, B_V is a locally convex A_U-module, so that the natural isomorphism $B_U \to B_V$ is continuous. Since B_V is Fréchet by 8.7, and B_U is Fréchet by 8.9, we conclude that it is a homeomorphism.

Finally, we have

$$M_U \; = \; M \otimes_A A_U \; = \; M \otimes_B B \otimes_A A_U \; = \; M \otimes_B B_U \; = \; M \otimes_B B_V \; = \; M_V \; .$$

□

Corollary 8.13. *Let* A *be a differentiable algebra and let* M *be a Fréchet* A*-module. If there exists some open cover* $\{U_i\}$ *of* $\mathrm{Spec}_r A$ *such that the* A_{U_i}*-modules* M_{U_i} *are free, with bounded rank, then* M *is a finitely generated projective* A*-module.*

Proof. In such a case $\widetilde{M}$ is a locally free $\widetilde{A}$-module of bounded rank; hence $\Gamma(\mathrm{Spec}_r A, \widetilde{M}\,)$ is a finitely generated projective A-module by 4.16. Now, we have $M = \Gamma(\mathrm{Spec}_r A, \widetilde{M}\,)$ because M is Fréchet.

□

Example 8.14. Let $\mathcal{V}$ be a smooth manifold. The natural inclusion morphism $\mathcal{C}^\infty(\mathcal{V}) \hookrightarrow \mathcal{C}^r(\mathcal{V})$, $0 \leq r < \infty$, is continuous, so that $\mathcal{C}^r(\mathcal{V})$ is a Fréchet $\mathcal{C}^\infty(\mathcal{V})$-module. The sheaf associated to $\mathcal{C}^r(\mathcal{V})$ is just the sheaf of real-valued functions of class $\mathcal{C}^r$ on $\mathcal{V} = \mathrm{Spec}_r \mathcal{C}^\infty(\mathcal{V})$. According to the Localization theorem for Fréchet modules, any function of class $\mathcal{C}^r$ on an open set $U \subset \mathcal{V}$ is a quotient of a function of class $\mathcal{C}^r$ by a function of class $\mathcal{C}^\infty$ without zeros in U, both defined on $\mathcal{V}$.

On the other hand, the $\mathcal{C}^\infty(\mathcal{V})$-module $\mathcal{T}^p_q(\mathcal{V})$ of all $\mathcal{C}^\infty$-differentiable tensor fields of type (p, q) on $\mathcal{V}$ is Fréchet, hence any $\mathcal{C}^\infty$-differentiable tensor field of type (p, q) on an open set $U \subset \mathcal{V}$ is a quotient of a $\mathcal{C}^\infty$-differentiable tensor field of type (p, q) by a $\mathcal{C}^\infty$-differentiable function without zeros in U, both defined on $\mathcal{V}$.

9 Finite Morphisms

Finite morphisms of differentiable spaces are the natural context for Malgrange's preparation theorem. A morphism of differentiable spaces $\varphi\colon Y \to X$ is defined to be finite if it is closed and its fibres $\varphi^{-1}(x)$ are finite differentiable spaces of bounded degree. The preparation theorem provides a characterization of finite morphisms $\varphi\colon \operatorname{Spec}_r B \to \operatorname{Spec}_r A$ between affine differentiable spaces: φ is finite if only if $\varphi^*\colon A \to B$ is a finite morphism of algebras, i.e., B is finitely generated as an A-module.

We also consider finite flat morphisms of differentiable spaces, which are the differentiable version of the ramified covering maps between topological spaces. Finite flat morphisms have nice properties: they are open and closed maps with fibres of locally constant degree, and they admit a reasonable notion of index of ramification.

9.1 Finite Differentiable Spaces

Definition. An $\mathbb{R}$-algebra A of finite dimension as a real vector space is said to be a **finite $\mathbb{R}$-algebra**. The dimension of A is said to be the **degree** of A.

In general, a morphism of rings $A \to B$ is said to be **finite** (we also say that B is a finite A-algebra) when B is a finitely generated A-module:

$$B \;=\; Ab_1 + \ldots + Ab_d \;.$$

A finite $\mathbb{R}$-algebra A is said to be **rational** when every maximal ideal of A is a real ideal. Rational finite $\mathbb{R}$-algebras are usually named **Weil algebras**.

Lemma 9.1. *Any quotient algebra of* $J^r_n := \mathbb{R}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^{r+1}$ *is a differentiable algebra.*

Proof. According 1.9 and 1.11 we have

$$J^r_n = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{m}_0^{r+1}$$

where $\mathfrak{m}_0^{r+1}$ is the closed ideal of all differentiable functions f on $\mathbb{R}^n$ whose Taylor expansion of order r at the origin vanishes: $j^r_0 f = 0$. Hence J^r_n is a differentiable algebra.

Since J^r_n is of finite dimension as a real vector space, we have that any ideal of J^r_n is closed. We conclude by 2.25.

□

J.A. Navarro González and J.B. Sancho de Salas: LNM 1824, pp. 99–111, 2003.

Theorem 9.2. *A finite $\mathbb{R}$-algebra A is a differentiable algebra if and only if it is rational.*

Proof. Let us assume that A is a differentiable algebra. Since A is finite, any residue field $k = A/\mathfrak{m}$ is a finite extension of $\mathbb{R}$, hence $k = \mathbb{R}$ or $k = \mathbb{C}$. If $k = \mathbb{C}$ then there exists $a \in A$ such that $a^2 + 1 \in \mathfrak{m}$. It is clear that $a^2 + 1$ does not vanish at any point of $\mathrm{Spec}_r A$, hence $a^2 + 1$ is an invertible element of A (see 2.15) so contradicting that $a^2 + 1 \in \mathfrak{m}$. We conclude that A is rational.

Conversely, let us show that any rational finite $\mathbb{R}$-algebra A is a differentiable algebra. In fact, any finite $\mathbb{R}$-algebra is a direct product of local finite algebras (see [1] 8.7), so that we may assume that A has a unique maximal ideal $\mathfrak{m}$. Since A is a finite $\mathbb{R}$-algebra, we have $\mathfrak{m}^{r+1} = \mathfrak{m}^{r+2}$ for some exponent r and Nakayama's lemma let us obtain that $\mathfrak{m}^{r+1} = 0$. Now, let $\{f_1, \ldots, f_n\}$ be a basis of $\mathfrak{m}$ as a vector space. Then the morphism of $\mathbb{R}$-algebras

$$\phi\colon \mathbb{R}[x_1, \ldots, x_n] \longrightarrow A \quad , \quad \phi(x_i) = f_i \ ,$$

is surjective. Since $\mathfrak{m}^{r+1} = 0$, the above morphism induces a surjective morphism

$$\phi\colon \mathbb{R}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^{r+1} \longrightarrow A \quad , \quad \phi(x_i) = f_i \ ,$$

Now 9.1 let us conclude that A is a differentiable algebra.

□

Proposition 9.3. *Let X be a differentiable space. The following conditions are equivalent:*

1. *X is the real spectrum of a rational finite $\mathbb{R}$-algebra.*
2. *X is a finite topological space and the $\mathbb{R}$-algebra of global differentiable functions $\mathcal{O}_X(X)$ is finite.*
3. *X is a finite topological space and the stalk $\mathcal{O}_{X,x}$ at any point $x \in X$ is a finite $\mathbb{R}$-algebra.*

In such a case, $\mathcal{O}_X(X) = \mathcal{O}_{X,x_1} \oplus \ldots \oplus \mathcal{O}_{X,x_r}$ *, where* $X = \{x_1, \ldots, x_r\}$.

Proof. ($1 \Rightarrow 2$) Let $X = \mathrm{Spec}_r A$ where A is a rational finite $\mathbb{R}$-algebra. The set $\mathrm{Spec}_r A$ is finite, because any family $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of maximal ideals of A defines a strictly increasing sequence of vector subspaces of A:

$$(\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r) \subset (\mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_r) \subset \ldots \subset \mathfrak{m}_r \subset A \ ,$$

so that r is bounded by the degree of A. Since $\mathcal{O}_X(X) = \widetilde{A}\,(X) = A$ (by the Localization theorem for differentiable algebras), we conclude that $\mathcal{O}_X(X)$ is a finite $\mathbb{R}$-algebra.

($2 \Leftrightarrow 3$) If X is finite, $X = \{x_1, \ldots, x_r\}$, then it is discrete, since any point of a differentiable space is closed. Hence

$$\mathcal{O}_X(X) = \mathcal{O}_X(x_1) \oplus \ldots \oplus \mathcal{O}_X(x_r) = \mathcal{O}_{X,x_1} \oplus \ldots \oplus \mathcal{O}_{X,x_r}$$

and the $\mathbb{R}$-algebra $\mathcal{O}_X(X)$ is finite if and only if so are all the rings of germs $\mathcal{O}_{X,x_1}, \dots, \mathcal{O}_{X,x_r}$.

$(2 \Rightarrow 1)$ X is affine, since any finite disjoint union of affine open sets is affine (recall that any finite direct product of differentiable algebras is a differentiable algebra; see 2.32). Then $\mathcal{O}_X(X)$ is a finite differentiable algebra, hence it is rational by 9.2.

□

Definition. A differentiable space X is said to be **finite** if it satisfies the equivalent conditions of 9.3. In such a case, we say that the degree of the finite $\mathbb{R}$-algebra $\mathcal{O}_X(X)$ is the **degree** of X.

Definition. Let $\varphi\colon Y = \operatorname{Spec}_r B \longrightarrow X = \operatorname{Spec}_r A$ be a morphism of affine differentiable spaces and let $x \in X$. The fibre $\varphi^{-1}(x) = x \times_X Y$ is a closed differentiable subspace of Y. According 7.3, $\varphi^{-1}(x)$ is an affine differentiable space defined by the following differentiable algebra

$$(A/\mathfrak{m}_x)\widehat{\otimes}_A B \;=\; \widehat{B/\mathfrak{m}_x B} \;=\; \widehat{B}/\widehat{\mathfrak{m}_x B} \;=\; B/\,\overline{\mathfrak{m}_x B}\ ,$$

i.e.,

$$\varphi^{-1}(x) \;=\; \operatorname{Spec}_r(B/\,\overline{\mathfrak{m}_x B}\,)\ .$$

Therefore, we say that $B/\,\overline{\mathfrak{m}_x B}$ is the **ring of the fibre** of x.

Lemma 9.4. *Let I be a finitely generated ideal of a differentiable algebra A. If $A/\,\overline{I}$ is a finite algebra, then I is a closed ideal.*

Proof. Let $x \in \operatorname{Spec}_r A$ and let $\mathfrak{m}_x$ be the maximal ideal of the local ring A_x. Since $A_x/\,\overline{I}_x = (A/\overline{I})_x$ is a local finite algebra, we have $\mathfrak{m}_x^r \subseteq \overline{I}_x$ for some exponent $r \geq 1$. Since $\mathfrak{m}_x^r$ is a finitely generated ideal and $\overline{I}_x/\mathfrak{m}_x^r \subset A_x/\mathfrak{m}_x^r$ is a finite dimensional vector space, we obtain that $\overline{I}_x$ is a finitely generated ideal. In virtue of the Spectral theorem 2.28, we have

$$I_x(A_x/\mathfrak{m}_x^{r+1}) \;=\; \overline{I}_x(A_x/\mathfrak{m}_x^{r+1}) \;=\; \overline{I}_x/\mathfrak{m}_x^{r+1}\ ,$$

hence the composition morphism

$$I_x \;\longrightarrow\; \overline{I}_x/\mathfrak{m}_x^{r+1} \;\longrightarrow\; \overline{I}_x/\mathfrak{m}_x\overline{I}_x$$

is surjective. By a standard application of Nakayama's lemma, we obtain that $\overline{I}_x$ is generated by elements of I_x. That is to say, $I_x = \overline{I}_x$ and then $\widetilde{I} = (\overline{I}\,)^{\sim}$. Now, 3.10 and 3.13 let us conclude that

$$I \;=\; \Gamma(\operatorname{Spec}_r A, \widetilde{I}) \;=\; \Gamma(\operatorname{Spec}_r A, (\overline{I})^{\sim}) \;=\; \overline{I}\ .$$

□

Proposition 9.5. *Let* $\varphi\colon \operatorname{Spec}_r B \to \operatorname{Spec}_r A$ *be a morphism of affine differentiable spaces and let* $x \in \operatorname{Spec}_r A$*. The ring of the fibre* $B/\overline{\mathfrak{m}_x B}$ *is finite if and only if so is the algebraic ring of the fibre* $B/\mathfrak{m}_x B$*. In such a case the ideal* $\mathfrak{m}_x B$ *is closed in* B *and*

$$B/\overline{\mathfrak{m}_x B} \;=\; B/\mathfrak{m}_x B \;=\; (B_{y_1}/\mathfrak{m}_x B_{y_1}) \oplus \ldots \oplus (B_{y_r}/\mathfrak{m}_x B_{y_r})$$

where $y_1, \ldots, y_r$ *are the points of the fibre* $\varphi^{-1}(x)$*.*

Proof. The ideal $\mathfrak{m}_x B$ is finitely generated; hence, if $B/\overline{\mathfrak{m}_x B}$ is a finite algebra, then $\mathfrak{m}_x B$ is closed by 9.4, and $B/\overline{\mathfrak{m}_x B} = B/\mathfrak{m}_x B$.

Moreover, $\operatorname{Spec}_r (B/\overline{\mathfrak{m}_x B}\,) = \varphi^{-1}(x) = \{y_1, \ldots, y_r\}$, and 9.3 let us conclude that

$$B/\mathfrak{m}_x B \;=\; (B/\mathfrak{m}_x B)_{y_1} \oplus \ldots \oplus (B/\mathfrak{m}_x B)_{y_r} \;.$$

□

Example. Let $f(t)$ be a differentiable function on $\mathbb{R}$ with a finite number of zeros $a_1, \ldots, a_r \in \mathbb{R}$, where $f(t)$ has finite order (in the sense that some derivative $f^{n_i)}(a_i) \neq 0$ for each index $1 \leq i \leq r$). Then we have $f\mathcal{O}_{a_i} = ((t - a_i)^{n_i})$, so that the ring of the fibre $f(t) = 0$ is finite, of degree $n_1 + \ldots + n_r$. Hence (f) is a closed ideal and

$$\mathcal{C}^\infty(\mathbb{R})/(f) \;\simeq\; \mathbb{R}[\,t\,]/(t - a_1)^{n_1} \oplus \ldots \oplus \mathbb{R}[\,t\,]/(t - a_r)^{n_r} \;.$$

9.2 Finite Morphisms

Definition. A morphism of differentiable spaces $\varphi\colon Y \to X$ is said to be **finite** if it is a closed separated map and for any $x \in X$ the fibre $\varphi^{-1}(x)$ is a finite differentiable space of bounded degree. We say that the degree of $\varphi^{-1}(x)$ is the **degree** of the morphism φ at the point $x \in X$.

Proposition 9.6. *Let* $\varphi\colon Y \to X$ *be a closed separated continuous map. If the fibre of* $x \in X$ *is a finite set,* $\varphi^{-1}(x) = \{y_1, \ldots, y_r\}$*, then* x *has a basis of open neighbourhoods* U *such that* $\varphi^{-1}(U)$ *is a disjoint union*

$$\varphi^{-1}(U) \;=\; V_1 \coprod \ldots \coprod V_r$$

of neighbourhoods $V_1, \ldots, V_r$ *of* $y_1, \ldots, y_r$ *respectively. Moreover, such neighbourhoods* V_i *define a basis of neighbourhoods of* y_i *in* Y *for any* $i = 1, \ldots, r$*.*

Proof. Since φ is a separated map, there exist disjoint open neighbourhoods $V_1, \ldots, V_r$ of $y_1, \ldots, y_r$ in Y. If U is an open neighbourhood of x which does not intersect the closed set $\varphi\left((V_1 \cup \ldots \cup V_r)^c\right)$, then

$$\varphi^{-1}(U) \;=\; (V_1 \cap \varphi^{-1}(U)) \coprod \ldots \coprod (V_r \cap \varphi^{-1}(U))$$

and any other open neighbourhood of x contained in U also satisfies the required condition. Moreover, since $(V_i \cap \varphi^{-1}(U)) \subseteq V_i$, these neighbourhoods of y_i define a basis of neighbourhoods of y_i.

□

Lemma 9.7. *Let* $\varphi\colon \operatorname{Spec}_r B \to \operatorname{Spec}_r A$ *be a morphism of affine differentiable spaces. If* φ *is a closed map and the fibre of a point* $x \in \operatorname{Spec}_r A$ *is a finite set,* $\varphi^{-1}(x) = \{y_1, \ldots, y_r\}$, *then*

$$B_x \;=\; B_{y_1} \oplus \ldots \oplus B_{y_r} \ .$$

Proof. By definition B_x is the localization of B by the differentiable functions $f \in A$ which do not vanish at x:

$$B_x \;=\; \varinjlim_{x\in U} B_U \ ,$$

where U is an arbitrary open neighbourhood of x in $\operatorname{Spec}_r A$. Now, using that $B_U = B_{\varphi^{-1}U}$ (by 8.12) and the Localization theorem, we have

$$B_x \;=\; \varinjlim_{x\in U} B_{\varphi^{-1}U} \;=\; \varinjlim_{x\in U} \widetilde{B}\,(\varphi^{-1}U) \ .$$

Since φ is a closed separated continuous map, 9.6 let us conclude that, restricting U to a basis of open neighbourhoods of x, we have

$$B_x \;=\; \varinjlim_{x\in U} \left(\widetilde{B}\,(V_1) \oplus \ldots \oplus \widetilde{B}\,(V_r)\right) \;=\; B_{y_1} \oplus \ldots \oplus B_{y_r} \ .$$

□

Preparation Theorem (Malgrange [26]). *Let* $\varphi\colon Y \to X$ *be a morphism of differentiable spaces, let* y *be a point of* Y *and let* $x = \varphi(y) \in X$. *The morphism* $\varphi^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ *is finite if and only if* $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ *is a finite* $\mathbb{R}$*-algebra.*

Theorem 9.8 ([13]). *A morphism of differentiable algebras* $j\colon A \to B$ *is finite if and only if* $j^*\colon \operatorname{Spec}_r B \to \operatorname{Spec}_r A$ *is a finite morphism of differentiable spaces.*

Proof. Let us assume that $A \to B$ is finite: $B = Ab_1 + \ldots + Ab_d$. For any $x \in \operatorname{Spec}_r A$, we have that $B/\mathfrak{m}_x B = (A/\mathfrak{m}_x)\bar{b}_1 + \ldots + (A/\mathfrak{m}_x)\bar{b}_d$ is a finite algebra over $A/\mathfrak{m}_x = \mathbb{R}$ of degree $\leq d$. Hence the fibre $j^{*\,-1}(x) = \operatorname{Spec}_r B/\overline{\mathfrak{m}_x B}$ is a finite differentiable space of degree $\leq d$.

It is clear that j^* is a separated map since $\operatorname{Spec}_r B$ is a separated (i.e. Hausdorff) space. Let us prove that j^* is a closed map. If $C \subseteq \operatorname{Spec}_r B$ is a closed set, then there exists a closed ideal $\mathfrak{b} \subset B$ such that $C = (\mathfrak{b})_0$. Let $\mathfrak{a} = j^{-1}(\mathfrak{b})$. It is enough to show that the map

$$j^*\colon C \;=\; (\mathfrak{b})_0 \;=\; \operatorname{Spec}_r (B/\mathfrak{b}) \;\longrightarrow\; \operatorname{Spec}_r (A/\mathfrak{a}) \;=\; (\mathfrak{a})_0$$

is surjective. Since the morphism $j\colon A/\mathfrak{a} \to B/\mathfrak{b}$ is finite and injective, we have to prove that $j^*\colon \operatorname{Spec}_r B \to \operatorname{Spec}_r A$ is surjective whenever $j\colon A \to B$ is finite and injective. In such a case $j_x\colon A_x \to B_x$ is injective for any $x \in \operatorname{Spec}_r A$; hence $B_x \neq 0$ and Nakayama's lemma shows that $\mathfrak{m}_x B_x \neq B_x$, so that $\mathfrak{m}_x B \neq B$. Now 9.5 states that $B/\overline{\mathfrak{m}_x B} = B/\mathfrak{m}_x B \neq 0$ and we conclude that the fibre $\varphi^{-1}(x) = \operatorname{Spec}_r (B/\mathfrak{m}_x B)$ is not empty by 2.18.

Conversely, let $x \in \operatorname{Spec}_r A$ and let ${j^*}^{-1}(x) = \{y_1, \dots, y_r\}$. By hypothesis $B_{y_i}/\mathfrak{m}_x B_{y_i}$ is a finite algebra, and the preparation theorem let us conclude that B_{y_i} is a finite A_x-algebra. Hence $B_x = B_{y_1} \oplus \dots \oplus B_{y_r}$ is a finite A_x-algebra.

Moreover, there exists a constant d such that $\dim B/\mathfrak{m}_x B \le d$ for any $x \in \operatorname{Spec}_r A$. If $B = \mathcal{C}^\infty(\mathbb{R}^m)/\mathfrak{b}$, then the monomials in $t_1, \dots, t_m$ generate $B/\mathfrak{m}_x B$ because they generate a dense vector subspace in $\mathcal{C}^\infty(\mathbb{R}^m)$. Since $1, t_i, t_i^2, \dots, t_i^d$ are linearly dependent in $B/\mathfrak{m}_x B$, it follows that any monomial is a linear combination of

$$t_1^{a_1} \cdot \ldots \cdot t_m^{a_m} \quad , \qquad 0 \le a_1, \dots, a_m < d \, ,$$

so that these monomials generate $B/\mathfrak{m}_x B$ for any $x \in \operatorname{Spec}_r A$, and Nakayama's lemma let us conclude that they generate B_x as A_x-module. Therefore, these monomials define an epimorphism of $\widetilde{A}$-modules $(\widetilde{A})^r \to \widetilde{B}$ and, taking global sections, the Localization theorem for Fréchet modules show that the corresponding morphism $A^r \to B$ is surjective. That is to say, such monomials generate the A-module B. The morphism $A \to B$ is finite.

□

Theorem 9.9. *Let $\varphi\colon Y \to X$ be a finite morphism of differentiable spaces. If X is affine then so is Y.*

Proof. By the Embedding theorem, it is enough to show that Y is separated, of bounded embedding dimension, and that the topology of Y has a countable basis. Y is separated since so are the morphism φ and the affine differentiable space $\operatorname{Spec}_r A$. Moreover, the topology of Y admits a countable basis because so does the topology of $\operatorname{Spec}_r A$ and, φ being a closed map with finite fibres, we may apply 9.6. Finally, the embedding dimension of Y at any point $y \in Y$ is bounded by the sum of the embedding dimension of $\operatorname{Spec}_r A$ at $x = \varphi(y)$ with the degree of φ at x. In fact, to prove it we may assume that Y is affine, $Y = \operatorname{Spec}_r B$, and in such a case we have the following exact sequence:

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \mathfrak{m}_y/\mathfrak{m}_y^2 \longrightarrow \overline{\mathfrak{m}}_y/\overline{\mathfrak{m}}_y^2 \longrightarrow 0$$

where $\overline{\mathfrak{m}}_y$ denotes the image of $\mathfrak{m}_y$ in the ring of the fibre $\overline{B} = B/\mathfrak{m}_x B$. We conclude because $\dim(\overline{\mathfrak{m}}_y/\overline{\mathfrak{m}}_y^2) < \dim \overline{B} \le$ degree of φ at x. Let us prove the exactness of the above sequence: the morphism $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \overline{\mathfrak{m}}_y/\overline{\mathfrak{m}}_y^2$ is obviously surjective, and its kernel is just $(\mathfrak{m}_x B + \mathfrak{m}_y^2)/\mathfrak{m}_y^2 = \mathfrak{m}_x B/(\mathfrak{m}_x B \cap \mathfrak{m}_y^2)$. Now, if $f_1, \dots, f_m \in B$ and $g_1, \dots, g_m \in A$, then, modulo $\mathfrak{m}_x B \cap \mathfrak{m}_y^2$, we have

$$\sum_{i=1}^m f_i \Delta_x(g_i) = \sum_{i=1}^m (f_i(y) + \Delta_y(f_i))\, \Delta_x(g_i) \equiv \sum_{i=1}^m f_i(y) \Delta_x(g_i)$$

□

9.3 Finite Flat Morphisms

Definition. A morphism of differentiable spaces $\varphi\colon Y \to X$ is said to be **flat** if the morphism $\varphi^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat for any $y \in Y$ and $x = \varphi(y) \in X$.

Proposition 9.10. *Any flat morphism of differentiable spaces $\varphi\colon Y \to X$ is an open map.*

Proof. Since the problem is local we may assume that X and Y are affine. Given a point $y \in Y$, let K be a compact neighbourhood of y. We have to show that the compact $\varphi(K)$ is a neighbourhood of $x = \varphi(y)$. If $\varphi(K)$ is not a neighbourhood of x, then there exists a sequence $x_n \to x$ where $x_n \notin \varphi(K)$. Let V_n be an open neighbourhood of x_n such that $V_n \cap \varphi(K) = \emptyset$ and $x_m \notin V_n$ for any $m \neq n$. Let $a_n \in \mathcal{O}(X)$ such that $a_n(x_n) = 1$ and $\operatorname{Supp} a_n \subseteq V_n$. Let

$$a = \sum_n \frac{a_n}{2^n q_n(a_n)}$$

where $\{q_1 \leq q_2 \leq \cdots\}$ is a fundamental system of seminorms of the differentiable algebra $\mathcal{O}(X)$. Note that

$$a(x_n) = \frac{a_n(x_n)}{2^n q_n(a_n)} \neq 0$$

hence $a_x \neq 0$. On the other hand, the Taylor expansion of a at any point $\bar{x}$ of $\varphi(K)$ vanishes,

$$j_{\bar{x}} a = \sum_n \frac{j_{\bar{x}} a_n}{2^n q_n(a_n)} = 0 \ ,$$

hence $j_{\bar{y}}\varphi^*(a) = \varphi^*(j_{\bar{x}}a) = 0$ for any $\bar{y} \in K$ and, K being a neighbourhood of y, we conclude that $\varphi^*(a)_y = 0$. Applying the functor $\otimes_{\mathcal{O}_x}\mathcal{O}_y$ to the exact sequence of $\mathcal{O}_x$-modules

$$0 \longrightarrow a_x\mathcal{O}_x \longrightarrow \mathcal{O}_x$$

we obtain an exact sequence because $\mathcal{O}_y$ is assumed to be a flat $\mathcal{O}_x$-module

$$0 \longrightarrow (a_x\mathcal{O}_x) \otimes_{\mathcal{O}_x} \mathcal{O}_y \longrightarrow \mathcal{O}_y$$

Therefore we have $(a_x\mathcal{O}_x) \otimes_{\mathcal{O}_x} \mathcal{O}_y = 0$, because $\varphi^*(a)_y = 0$. Now, we have $a_x\mathcal{O}_x \simeq \mathcal{O}_x/I$, where $I = \{b_x \in \mathcal{O}_x\colon\ a_x b_x = 0\}$, hence

$$0 = (a_x\mathcal{O}_x) \otimes_{\mathcal{O}_x} \mathcal{O}_y \simeq \mathcal{O}_y/I\mathcal{O}_y$$

and it follows that some germ $b_x \in I$ is invertible in $\mathcal{O}_y$. Since $\varphi^*(\mathfrak{m}_x) \subseteq \mathfrak{m}_y$, we conclude that b_x is invertible in $\mathcal{O}_x$, so contradicting that $a_x \neq 0$.

□

Lemma 9.11. *Let A be a differentiable algebra and let M be a closed submodule of a free A-module A^r of finite rank. If $M_x = 0$ for certain point $x \in \operatorname{Spec}_r A$, then $M_U = 0$ for some open neighbourhood U of x.*

Proof. Given $m = (a_1, \dots, a_r) \in M_U \subseteq A_U \oplus \cdots \oplus A_U$ and a point $p \in U$, we write $j_p m := (j_p a_1, \dots, j_p a_r)$. Note that $m = 0$ if and only if $j_p m = 0$ for any $p \in U$ (by 2.17, applied to A_U).

Now, if $M_U \neq 0$ for any open neighbourhood U of x, then there exists a sequence $x_i \to x$ and elements $m_i \in M$ such that $j_{x_i} m_i \neq 0$. Moreover, multiplying m_i by an adequate factor, we may assume that $\operatorname{Supp} m_i$ does not contain any point x_j for $j \neq i$. Now, let

$$m := \sum_i \frac{m_i}{2^i q_i(m_i)}$$

where $\{q_1 \leq q_2 \leq \cdots\}$ is a fundamental system of seminorms of M. Note that

$$j_{x_i} m = \frac{j_{x_i} m_i}{2^i q_i(m_i)} \neq 0,$$

hence $m_x \neq 0$, so contradicting that $M_x = 0$.

□

Corollary 9.12. *Let $\varphi\colon Y \to X$ be a morphism of differentiable spaces and let $y \in Y$, $x = \varphi(y) \in X$. If $\varphi^*\colon \mathcal{O}_x \to \mathcal{O}_y$ is an isomorphism, then φ is a local isomorphism at y.*

Proof. It is clear that $\varphi_*\colon T_yY \to T_xX$ is an isomorphism. By 5.21, φ is a local embedding at y, hence we may assume that $\varphi\colon Y = \operatorname{Spec}_r B \to \operatorname{Spec}_r A = X$ is a morphism of affine differentiable spaces defined by a surjective morphism $\varphi^*\colon A \to B$. Let $\mathfrak{a}$ be the kernel of $\varphi^*\colon A \to B$. Since $\varphi^*\colon A_x \to B_y = B_x$ is injective, we have that $\mathfrak{a}_x = 0$. By 9.11, we may assume that $\mathfrak{a} = 0$ and we conclude that $\varphi^*\colon A \to B$ is an isomorphism.

□

Proposition 9.13 ([41]). *Let A be a differentiable algebra and let M be a Fréchet A-module. If M is finitely generated then the following conditions are equivalent:*

1. *M is a flat A-module.*
2. *M_x is a flat A_x-module for any $x \in \operatorname{Spec}_r A$.*
3. *For each $x \in \operatorname{Spec}_r A$ there exists an open neighbourhood U such that M_U is a free A_U-module.*
4. *M is a projective A-module.*

   If in addition A is reduced then the above conditions are equivalent to the following one:
5. *The function $d\colon \operatorname{Spec}_r A \to \mathbb{N}$, $d(x) = \dim (M/\mathfrak{m}_x M)$, is locally constant.*

Proof. (2) ⇒ (3). Any flat finitely generated module over a local ring is free, hence $M_x = A_x \oplus \cdots \oplus A_x$. Let $m_1, \dots, m_r \in M$ such that $\{(m_1)_x, \dots, (m_r)_x\}$ is a basis of M_x. Since M is finitely generated, there exists an open neighbourhood V of x such that $m_1, \dots, m_r$ generate M_V as an A_V-module. Let us consider the morphism $\varphi\colon A \oplus \cdots \oplus A \to M$, $\oplus a_i \mapsto \sum a_i m_i$ and let $N = \operatorname{Ker}\varphi$. It is clear that $N_x = 0$, so that there exists an open neighbourhood W of x such that $N_W = 0$ by 9.11. Taking $U = V \cap W$ we obtain that $M_U = A_U \oplus \cdots \oplus A_U$.

(3) $\Rightarrow$ (4). We have that $\widetilde{M}$ is a locally free $\widetilde{A}$-module (of bounded rank because M is finitely generated) and we conclude that M is projective by 4.16.

(1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) are standard results of commutative algebra.

(5) $\Rightarrow$ (3). Given $x \in \operatorname{Spec}_r A$, let $r = d(x)$. Let $m_1, \ldots, m_r \in M$ such that $\{[m_1], \ldots, [m_r]\}$ is a basis of $M/\mathfrak{m}_x M$. By Nakayama's lemma, M_x is generated as an A_x-module by $\{(m_1)_x, \ldots, (m_r)_x\}$. Since M is finitely generated, there exists an open neighbourhood U of x such that M_U is generated by $m_1, \ldots, m_r$ as an A_U-module. Moreover, we may assume that d is constant on U, i.e., $d(z) = r$ for any $z \in U$; hence the classes $[m_1], \ldots, [m_r]$ define a basis of $M/\mathfrak{m}_z M$. Let us show that $\{m_1, \ldots, m_r\}$ is a basis of M_U, i.e., M_U is a free A_U-module. If $\sum a_i m_i = 0$ for some $a_i \in A_U$, then $[\sum a_i m_i] = \sum a_i(z)[m_i] = 0$ in $M/\mathfrak{m}_z M$ for any $z \in U$, hence $a_i(z) = 0$ for any $z \in U$. Since A_U is reduced, we may conclude that $a_i = 0$.

Finally, (3) $\Rightarrow$ (5) is obvious.

□

Corollary 9.14. *Let $\varphi\colon \operatorname{Spec}_r B \to \operatorname{Spec}_r A$ be a finite morphism of affine differentiable spaces. Then φ is a flat morphism if and only if $\varphi^*\colon A \to B$ is a flat morphism.*

Proof. By 9.13, B is a flat A-module if and only if B_x is a flat A_x-module for any $x \in \operatorname{Spec}_r A$. By 9.7, $B_x = B_{y_1} \oplus \ldots \oplus B_{y_r}$, where $\varphi^{-1}(x) = \{y_1, \ldots, y_r\}$, hence B_x is a flat A_x-module if and only if B_{y_i} is a flat A_x-module for any $y_i \in \varphi^{-1}(x)$.
□

Corollary 9.15. *Let $\varphi\colon Y \to X$ a finite morphism of differentiable spaces and let us assume that X is reduced. Then φ is a flat morphism if and only if the function $d(x) = \operatorname{degree}\{\varphi^{-1}(x)\}$ is locally constant on X.*

Proof. Since the question is local, we may assume that X is affine: $X = \operatorname{Spec}_r A$. By 9.9, Y also is affine: $Y = \operatorname{Spec}_r B$. We conclude by 9.14 and 9.13.

□

Definition. Let $\varphi\colon Y \to X$ be a finite flat morphism of differentiable spaces, let y be a point of Y and let $x = \varphi(y) \in X$. The **ramification index** of φ at y is defined to be

$$\operatorname{ind}_y\varphi := \dim\left(\mathcal{O}_y/\mathfrak{m}_x\mathcal{O}_y\right) .$$

Note that if $\deg_x\varphi$ denotes the degree of the finite differentiable space $\varphi^{-1}(x)$, then we have

$$\deg_x\varphi = \sum_{\varphi(y_i)=x} \operatorname{ind}_{y_i}\varphi .$$

Proposition 9.16. *Let $\varphi\colon Y \to X$ be a finite flat morphism of differentiable spaces. Let $y \in Y$ and $x = \varphi(y) \in X$. The following conditions are equivalent:*

1. *φ is a local isomorphism at $y \in Y$.*
2. $\mathrm{ind}_y \varphi = 1$.
3. *$\varphi_* \colon T_y Y \to T_x X$ is an isomorphism.*

Proof. We may assume that $\varphi\colon Y = \mathrm{Spec}_r\, B \to \mathrm{Spec}_r\, A = X$ is a morphism of affine differentiable spaces. By 9.7 we have

$$B_x \;=\; B_{y_1} \oplus \ldots \oplus B_{y_r}\ .$$

Since B_x is a finitely generated flat A_x-module, we have that B_{y_i} is a free A_x-module of rank $\dim (B_{y_i}/\mathfrak{m}_x B_{y_i}) = \mathrm{ind}_{y_i}\varphi$. Hence, $\varphi^*\colon A_x \to B_y$ is an isomorphism if and only if $\mathrm{ind}_y \varphi = 1$. Using 9.12 we conclude that (1) $\Leftrightarrow$ (2).

If $\varphi_*\colon T_y Y \to T_x X$ is an isomorphism, then φ is a local embedding at y (by 5.21), hence $\varphi^*\colon A_x \to B_y$ is surjective. Since B_y is a free A_x-module, we conclude that $\varphi^*\colon A_x \to B_y$ is an isomorphism. By 9.12, we obtain that φ is a local isomorphism at y.

Finally, the implication (1) $\Rightarrow$ (3) is obvious.

□

Definition. Let $\varphi\colon Y \to X$ be a finite flat morphism of differentiable spaces. A point $y \in Y$ is said to be a **ramification point** if φ is not a local isomorphism at y, i.e., $\mathrm{ind}_y \varphi > 1$.

The following result is useful to determine generators of a finite morphism $A \to B$ of differentiable algebras.

Lemma 9.17. *Let A be a differentiable algebra and let M be a finitely generated Fréchet A-module. Given $m_1, \ldots, m_r \in M$, we have*

(a) *If the classes $[m_1], \ldots, [m_r]$ generate $M/\mathfrak{m}_x M$ for any $x \in \mathrm{Spec}_r\, A$, then $m_1, \ldots, m_r$ generate $M = Am_1 + \ldots + Am_r$.*

(b) *If $\{[m_1], \ldots, [m_r]\}$ is a basis of $M/\mathfrak{m}_x M$ for any $x \in \mathrm{Spec}_r\, A$ and A is reduced, then M is a free A-module and $\{m_1, \ldots, m_r\}$ is a basis:*

$$M \;=\; A\, m_1 \oplus \ldots \oplus A\, m_r$$

Proof. (*a*) Let $\varphi\colon L = \bigoplus^r A \to M$, $\oplus a_i \mapsto \sum a_i m_i$. By Nakayama's lemma $(m_1)_x, \ldots (m_r)_x$ generate M_x as an A_x-module, hence $L_x \to M_x$ is surjective for any $x \in \mathrm{Spec}_r\, A$. Then $\varphi\colon\ \widetilde{L} \to \widetilde{M}$ is an epimorphism of sheaves of $\widetilde{A}$-modules, hence $\widetilde{L}\,(X) \to \widetilde{M}\,(X)$ is surjective. By the Localization theorem for Fréchet modules, we have that $\widetilde{L}\,(X) = L$ and $\widetilde{M}\,(X) = M$, hence $\varphi\colon L \to M$ is surjective, i.e., $m_1, \ldots, m_r$ generate M.

(*b*) If $\sum a_i m_i = 0$ for some $a_i \in A$, then $[\sum a_i m_i] = \sum a_i(x)[m_i] = 0$ in $M/\mathfrak{m}_x M$ for any $x \in \mathrm{Spec}_r\, A$, hence $a_i(x) = 0$ for any $x \in \mathrm{Spec}_r\, A$. Since A is reduced, we may conclude that $a_i = 0$.

□

9.4 Examples

Example 9.18. Let $P_1, \dots, P_r$ be polynomials in n variables. If

$$\mathbb{R}[x_1, \dots, x_n]/(P_1, \dots, P_r)$$

is a rational finite algebra, then the ideal $(P_1, \dots, P_r)$ of $\mathcal{C}^\infty(\mathbb{R}^n)$ is closed and

$$\mathcal{C}^\infty(\mathbb{R}^n)/(P_1, \dots, P_r) = \mathbb{R}[x_1, \dots, x_n]/(P_1, \dots, P_r) \ .$$

In fact, $\mathbb{R}[x_1, \dots, x_n]/(P_1, \dots, P_r)$ is a differentiable algebra by 9.2. Then 2.20 let us define an epimorphism

$$\varphi\colon \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}[x_1, \dots, x_n]/(P_1, \dots, P_r) \quad , \quad x_i \mapsto [x_i] \ ,$$

hence $\mathcal{C}^\infty(\mathbb{R}^n)/I = \mathbb{R}[x_1, \dots, x_n]/(P_1, \dots, P_r)$, where $I = \operatorname{Ker}\varphi$. Using Whitney's spectral theorem, it is easy to check that I is the closure of $(P_1, \dots, P_r)$. Now, 9.4 implies that $I = (P_1, \dots, P_r)$.

A similar argument proves the following more general result: Let $P_1, \dots, P_r$ be polynomials in n variables such that

$$A \ = \ \mathbb{R}[x_1, \dots, x_n]/(P_1, \dots, P_r)$$

is a finite algebra (not necessarily rational). Let

$$A \ = \ A_1 \oplus \cdots \oplus A_p \oplus B_1 \oplus \cdots \oplus B_q$$

be the decomposition of A as product of local algebras, where the algebras A_i are assumed to be rational while the algebras B_j are not rational. We define the rational finite algebra $A_{\text{rat}} := A_1 \oplus \cdots \oplus A_p$, which is a quotient of A. Then $(P_1, \dots, P_r)$ is a closed ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$ and

$$\mathcal{C}^\infty(\mathbb{R}^n)/(P_1, \dots, P_r) \ = \ A_{\text{rat}} \ .$$

More generally, if $f_1, \dots, f_r$ are real analytic functions on $\mathbb{R}^n$, then $(f_1, \dots, f_r)$ is a closed ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$ (see [26], Chapter VI, Th.1.1').

Example 9.19. Let $P(t)$ be a polynomial of degree n with real coefficients and let us consider the differentiable map $P\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto P(t)$. Let us show that it is a finite morphism with fibres of degree $\leq n$. It is closed because $P(t) \to \infty$ as $t \to \infty$. Given $a \in \mathbb{R}$, the fibre $P(t) = a$ is the finite set of real roots of $P(t) - a$. Let us write

$$P(t) - a = (t - a_1)^{n_1} \cdots (t - a_r)^{n_r} Q(t)$$

where $Q(t)$ is a polynomial without real roots, so that $Q(t)$ is an invertible element of $\mathcal{C}^\infty(\mathbb{R})$. By 9.18 the ring of the fibre $P(t) = a$ is

$$\begin{aligned} \mathcal{C}^\infty(\mathbb{R})/(P(t) - a) = \ & \mathcal{C}^\infty(\mathbb{R})/\big((t - a_1)^{n_1} \cdots (t - a_r)^{n_r}\big) \\ = \ & \mathbb{R}[\,t\,]/\big((t - a_1)^{n_1} \cdots (t - a_r)^{n_r}\big) \ , \end{aligned}$$

which is a finite algebra of degree $n_1 + \cdots + n_r \leq n$. In conclusion, $P\colon \mathbb{R} \to \mathbb{R}$ is a finite morphism. By 9.8, we know that $P^*\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ is a finite morphism. Using 9.17 it is easy to check that $1, t, \ldots, t^{n-1}$ is a system of generators of $P^*\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$. In particular, if we put $P(t) = t^n$ then we obtain that any differentiable function $f(t)$ on $\mathbb{R}$ admits a decomposition

$$f(t) \ = \ g_0(t^n) + g_1(t^n)t + \ldots + g_{n-1}(t^n)t^{n-1} \quad , \qquad g_i \in \mathcal{C}^\infty(\mathbb{R}) \ .$$

When $n = 2$, we obtain that any differentiable function $f(t)$ admits a decomposition $f(t) = g_0(t^2) + g_1(t^2)t$. Hence, if $f(t)$ is an even function, $f(t) = f(-t)$, then $2g_1(t^2)t = 0$ and $g_1(t^2) = 0$. We conclude that any even differentiable function is a differentiable function of t^2.

Example 9.20. Let us consider the circle C of equation $x^2 + y^2 - 1 = 0$ in $\mathbb{R}^2$. By 2.4 and 2.7, we have that $\mathcal{C}^\infty(C) = \mathcal{C}^\infty(\mathbb{R}^2)/(x^2 + y^2 - 1)$. Let us show that the projection $x\colon C \to [-1, 1]$ is a finite flat morphism. It is a closed map since C is compact. The ring of the fibre $x = a$ is

$$\mathcal{C}^\infty(\mathbb{R}^2)/(y^2 + x^2 - 1, x - a) \ = \ \mathcal{C}^\infty(\mathbb{R})/(y^2 + a^2 - 1) \ = \ \mathbb{R}[y]/(y^2 + a^2 - 1)$$

which is a finite algebra of degree 2. Therefore $x\colon C \to [-1, 1]$ is a finite morphism, and it is flat by 9.15.

Note that the projection $x\colon C \to [-1, 1]$ is ramified at $(-1, 0)$ and $(1, 0)$ with index of ramification 2.

Since the ring of each fibre is a finite algebra spanned (as a real vector space) by $\{1,\, y\}$, we conclude from 9.17.b that $\mathcal{C}^\infty(C)$ is a free $\mathcal{C}^\infty([-1, 1])$-module, $\{1,\, y\}$ being a basis. Therefore, any differentiable function $f(x, y)$ on the circle admits a decomposition

$$f(x, y) \ = \ g_1(x) + g_2(x)y \quad , \qquad g_1, g_2 \in \mathcal{C}^\infty([-1, 1]) \ .$$

In particular, if a differentiable function $f(x, y)$ on C is invariant by the symmetry $\sigma(x, y) = (x, -y)$, then $f(x, y) = g_1(x)$.

Example 9.21. Let us consider the elementary symmetric functions in n variables

$$\begin{aligned} s_1 &:= x_1 + \ldots + x_n \\ s_r &:= \sum_{1 \leq i_1 < \ldots < i_r \leq n} x_{i_1} \cdot \ldots \cdot x_{i_r} \\ s_n &:= x_1 \cdot \ldots \cdot x_n \end{aligned}$$

and note that $\mathbb{R}[x_1, \ldots, x_n]$ is a finite algebra over $A = \mathbb{R}[s_1, \ldots, s_n]$, generated as A-module by the $n!$ monomials

$$(*) \qquad x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \quad , \qquad 0 \leq a_i \leq n - i \ ,$$

because x_1 satisfies an equation of integral dependence

$$0 \ = \ x_1^n - s_1 x_1^{n-1} + s_2 x_1^{n-2} - \ldots + (-1)^n s_n$$

with coefficients in A, x_2 satisfies (note that $t^n - s_1 t^{n-1} + \ldots$ is a multiple of $t - x_1$ in $A[t]$, since it has the root $t = x_1$) an equation of integral dependence:

$$0 = \frac{x_2^n - s_1 x_2^{n-1} + \ldots + (-1)^n s_n}{x_2 - x_1} = x_2^{n-1} + (x_1 - s_1) x_2^{x-2} + \ldots$$

with coefficients in $A[x_1]$ and, in general, x_{i+1} satisfies an equation of integral dependence of degree $n - i$ with coefficients in $A[x_1, \ldots, x_i]$. In fact, even if we shall not use it, $\mathbb{R}[x_1, \ldots, x_n]$ is a free A-module of rank $n!$ and the monomials $(*)$ define a basis, because they are linearly independent over A. In fact $\mathbb{R}(x_1, \ldots, x_n)$ is a Galois extension of $\mathbb{R}(s_1, \ldots, s_n)$ of degree $n!$ spanned by the monomials $(*)$, so that they are linearly independent over $\mathbb{R}(s_1, \ldots, s_n)$, hence so they are over $A \subset \mathbb{R}(s_1, \ldots, s_n)$.

Therefore, if $c_1, \ldots, c_n \in \mathbb{R}$, then $\mathbb{R}[x_1, \ldots, x_n]/(s_1 - c_1, \ldots, s_n - c_n)$ is a finite $\mathbb{R}$-algebra of degree $n!$ which is spanned, as a real vector space, by the $n!$ monomials $(*)$.

Now, let us consider the differentiable map $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by the elementary symmetric functions, $\varphi(x_1, \ldots x_n) = (s_1, \ldots, s_n)$. By 9.18, the ring of the fibre of a point $c = (c_1, \ldots, c_n)$ is the finite algebra

$$\big[\,\mathbb{R}[x_1, \ldots, x_n]/(s_1 - c_1, \ldots, s_n - c_n)\,\big]_{\text{rat}} \,.$$

Moreover, φ is a closed map since any root α of a polynomial may be bounded by the coefficients:

$$|\alpha| \leq 1 + \max\{|c_1|, \ldots, |c_n|\} \,,$$

and we conclude that φ is a finite morphism.

By 9.8 and 9.17.a, we have that $\varphi^* : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ is a finite morphism and that it is generated by the $n!$ monomials $(*)$. Then any differentiable function $f(x_1, \ldots, x_n)$ admits a decomposition

$$f(x_1, \ldots, x_n) = \sum_{0 \leq a_i < n-i} g_{a_1 \ldots a_n}(s_1, \ldots, s_n) x_1^{a_1} \ldots x_n^{a_n} \,,$$

where $g_{a_1 \ldots a_n} \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Therefore, if $f(x_1, \ldots, x_n)$ is a symmetric function (= invariant under permutations of the variables), then $f(x_1, \ldots, x_n) = g(s_1, \ldots, s_n)$ for some differentiable function g. In fact, if a point $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ corresponds to a polynomial with n different real roots, then the fibre of φ over c has $n!$ different points and, since it is spanned by $n!$ elements as a vector space, we conclude that the ring of the fibre $\varphi^{-1}(c)$ is $\mathbb{R} \oplus \overset{n!}{\ldots} \oplus \mathbb{R}$ and that its invariant elements under arbitrary permutations are just the constants. Hence $g_{a_1 \ldots a_n}(s_1, \ldots, s_n)$ vanishes on the fibre of c whenever $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$. That is to say, $g_{a_1 \ldots a_n}$ vanishes on the points of $\mathbb{R}^n$ without two coincident coordinates and, by continuity, $g_{a_1 \ldots a_n} = 0$.

10 Smooth Morphisms

We introduce the sheaf of differentials Ω_X for any differentiable space X. In the case of a smooth manifold, this sheaf coincides with the sheaf of all differentiable 1-forms on X. The sheaf Ω_X is used to characterize when a differentiable space X is a smooth manifold or a Whitney space:

–. If the underlying topological space is a topological manifold of constant dimension n, then X is a smooth manifold if and only if Ω_X is a locally free $\mathcal{O}_X$-module of rank n.

–. Ω_X is a locally free $\mathcal{O}_X$-module if and only if any point of X has a neighbourhood isomorphic to the Whitney space of a closed subset Y of some $\mathbb{R}^n$.

10.1 Module of Relative Differentials

Let $k \to A$ be a morphism of rings (commutative and with 1). We shall always consider on $A \otimes_k A$ the structure of A-algebra induced by the natural morphism $A \to A \otimes_k A$, $a \mapsto 1 \otimes a$. The product of A defines a morphism of A-algebras $A \otimes_k A \to A$, whose kernel will be denoted by $\Delta_{A/k}$. It is easy to check that the ideal $\Delta_{A/k}$ is generated by the increments $\Delta a := a \otimes 1 - 1 \otimes a$, $a \in A$.

Definition. Let $k \to A$ be a morphism of rings. The module of **Kahler differentials** of A over k is defined to be $\mathfrak{D}_{A/k} := \Delta_{A/k}/\Delta^2_{A/k}$. It is a module over $(A \otimes_k A)/\Delta_{A/k} = A$.

It is easy to check that the natural map

$$\mathrm{d}\colon A \longrightarrow \mathfrak{D}_{A/k} \qquad , \qquad \mathrm{d}a := [\Delta a] = [a \otimes 1 - 1 \otimes a]$$

is a k-linear derivation: $\mathrm{d}(ab) = a(\mathrm{d}b) + b(\mathrm{d}a)$. Moreover, the A-module $\mathfrak{D}_{A/k}$ is generated by the elements $\mathrm{d}a$ because the ideal $\Delta_{A/k}$ is generated by the increments Δa.

Proposition 10.1. *Let $k \to A$ be a morphism of rings and let M be an A-module. Given a k-linear derivation $D\colon A \to M$, there exists a unique morphism of A-modules $\varphi\colon \mathfrak{D}_{A/k} \to M$ such that $D = \varphi \circ \mathrm{d}$.*

J.A. Navarro González and J.B. Sancho de Salas: LNM 1824, pp. 113–125, 2003.

Proof. It easy to check that $\tilde{D}\colon A\otimes_k A \to M$, $\tilde{D}(b\otimes a) := a\cdot Db$ is an A-linear derivation, where M is considered as an $A\otimes_k A$-module via the morphism $A\otimes_k A \dot{\to} A$. It is a general fact that, for any derivation $\bar{D}\colon A\to M$ and any ideal $I\subset A$, we have $\bar{D}(I^2)\subseteq IM$. In particular, $\tilde{D}(\Delta^2_{A/k})\subseteq \Delta_{A/k}M = 0$, hence $\tilde{D}$ induces a morphism of A-modules

$$\varphi\colon \mathfrak{D}_{A/k} = \Delta_{A/k}/\Delta^2_{A/k} \longrightarrow M$$

which has the desired property: $D = \varphi\circ\mathrm{d}$. Finally, the uniqueness of φ is obvious since $\mathfrak{D}_{A/k}$ is generated by the elements $\mathrm{d}a$.

□

Definition. Let $k\to A$ be a morphism of differentiable algebras. The kernel of the morphism $A\widehat{\otimes}_k A \dot{\to} A$ is a closed ideal denoted by $\mathcal{D}_{A/k}$ and it is said to be the **diagonal ideal**, since it is just the ideal of the diagonal embedding

$$\mathrm{Spec}_r\, A \hookrightarrow (\mathrm{Spec}_r\, A)\times_{\mathrm{Spec}_r k}(\mathrm{Spec}_r\, A).$$

Definition. The Fréchet A-module of **relative differentials** of A over k is defined to be

$$\Omega_{A/k} := \mathcal{D}_{A/k}/\overline{\mathcal{D}^2_{A/k}}\ .$$

The continuous k-linear derivation $\mathrm{d}\colon A\to\Omega_{A/k}$, $\mathrm{d}a := [\Delta a] = [a\otimes 1 - 1\otimes a]$, is said to be the **differential**. When $k=\mathbb{R}$, we say that $\Omega_{A/\mathbb{R}}$ is the **module of differentials** of A and we denote it by Ω_A.

In the more general setting of Fréchet algebras, modules of differentials were introduced and studied in [35], while modules of relative differentials and their properties were considered in [41]. Now a brief exposition in the realm of differentiable algebras follows.

Lemma 10.2. *Let $k\to A$ be a morphism of differentiable algebras. With the previous notations, we have: $\mathcal{D}_{A/k} = \widehat{\Delta}_{A/k}$.*

Proof. We have $A = (A\otimes_k A)/\Delta_{A/k}$. By 6.1.a, we obtain topological isomorphisms

$$A = \widehat{A} = [(A\otimes_k A)/\Delta_{A/k}]^\wedge = (A\widehat{\otimes}_k A)/\widehat{\Delta}_{A/k}$$

hence $\widehat{\Delta}_{A/k} = \mathcal{D}_{A/k}$.

□

Proposition 10.3. *Let $k\to A$ be a morphism of differentiable algebras. We have:*

$$\Omega_{A/k} = \widehat{\mathfrak{D}}_{A/k}\ .$$

Proof. By 6.1.b and 10.2, the cokernel of locally convex A-modules

$$\Delta^2_{A/k} \longrightarrow \Delta_{A/k} \longrightarrow \mathfrak{D}_{A/k} \longrightarrow 0$$

induces the cokernel of Fréchet A-modules

$$\widehat{\Delta^2_{A/k}} \longrightarrow \widehat{\Delta}_{A/k} = \mathcal{D}_{A/k} \longrightarrow \widehat{\mathfrak{D}}_{A/k} \longrightarrow 0 \ .$$

Each power I^r of any ideal I is dense in $\left(\overline{I}\right)^r$. In particular, $\Delta^2_{A/k}$ is dense in $\mathcal{D}^2_{A/k}$, and we obtain a topological isomorphism

$$\mathcal{D}_{A/k}/\overline{\mathcal{D}^2_{A/k}} \ = \ \widehat{\mathfrak{D}}_{A/k} \ .$$

□

Theorem 10.4. *Let $k \to A$ be a morphism of differentiable algebras and let M be a Fréchet A-module. Given a continuous k-linear derivation $D\colon A \to M$, there exists a unique morphism of Fréchet A-modules $\varphi\colon \Omega_{A/k} \to M$ such that $D = \varphi \circ \mathrm{d}$. That is to say,*

$$\mathrm{Der}_k(A, M) = \mathrm{Hom}_A(\Omega_{A/k}, M) \ .$$

Proof. It is a direct consequence of 10.1 and 10.3.

□

Proposition 10.5. *Let $k \to A$ be a morphism of differentiable algebras. If U is an open set in* $\mathrm{Spec}_r\, A$*, then we have a topological isomorphism*

$$\Omega_{A_U/k} \ = \ (\Omega_{A/k})_U \ .$$

Proof. The restriction morphism $A \to A_U$ induces a natural morphism of locally convex A-modules $\mathfrak{D}_{A/k} \to \mathfrak{D}_{A_U/k}$ and, by completion, we get a morphism $\Omega_{A/k} \to \Omega_{A_U/k}$. By the universal property of the localization, we obtain a morphism of locally convex A_U-modules

$$(\Omega_{A/k})_U \ \longrightarrow \ \Omega_{A_U/k} \ .$$

On the other hand, the k-linear derivation

$$D\colon A_U \ \longrightarrow \ (\Omega_{A/k})_U \quad , \quad D\left(\frac{a}{s}\right) = \frac{s\mathrm{d}a - a\mathrm{d}s}{s^2} \ ,$$

is continuous because, by 8.7, the topology of A_U is the final topology of the map $A \times S \to A_U$, $(a, s) \mapsto a/s$. Hence D induces a continuous morphism of A_U-modules

$$\Omega_{A_U/k} \ \longrightarrow \ (\Omega_{A/k})_U$$

which is just the inverse morphism.

□

Definition. Let $k \to A$ and $p\colon A \to B$ be morphisms of differentiable algebras. The kernel of the morphism $A\widehat{\otimes}_k B \to B$, $a \otimes b \mapsto p(a)b$ will be denoted by $\mathcal{D}_p$, and the **cotangent module** of A over k at the (parametrized) point p is defined to be the B-module

$$\Omega_p(A/k) \ := \ \mathcal{D}_p/\overline{\mathcal{D}^2_p} \ .$$

When p is the identity $A \to A$, by definition $\Omega_{id}(A/k) = \Omega_{A/k}$.

Let Δ_p be the kernel of the morphism $A \otimes_k B \to B$, $a \otimes b \mapsto p(a)b$. The same arguments used in 10.2 and 10.3 show that

$$\mathcal{D}_p \;=\; \widehat{\Delta}_p \qquad , \qquad \Omega_p(A/k) \;=\; [\Delta_p/\Delta_p^2]^{\wedge} \ .$$

Example 10.6. Let A be a differentiable algebra and let $x \in X = \operatorname{Spec}_r A$. The cotangent $\mathbb{R}$-module $\Omega_x(A/\mathbb{R})$ at the point $x\colon A \to A/\mathfrak{m}_x$ is

$$\Omega_x(A/\mathbb{R}) \;=\; \mathfrak{m}_x/\mathfrak{m}_x^2 \;\overset{5.12}{=}\; T_x^*X$$

because the kernel of the morphism $(x, Id)\colon A\widehat{\otimes}_{\mathbb{R}}(A/\mathfrak{m}_x) = A \to A/\mathfrak{m}_x$ is just $\mathfrak{m}_x$, and $\mathfrak{m}_x^2$ is closed in A.

Proposition 10.7. *Let $k \to A \xrightarrow{p} B$ be morphisms of differentiable algebras. We have a topological isomorphism*

$$\Omega_{A/k}\widehat{\otimes}_A B \;=\; \Omega_p(A/k) \ .$$

Proof. The following exact sequence of A-modules

$$0 \longrightarrow \Delta \longrightarrow A \otimes_k A \xrightarrow{(Id,Id)} A \longrightarrow 0$$

splits topologically (i.e., it admits a continuous section or retraction), since we have the continuous section $A \to A \otimes_k A$, $a \mapsto 1 \otimes a$. Applying the functor $\otimes_A B$ we obtain that the exact sequence

$$0 \longrightarrow \Delta \otimes_A B \longrightarrow A \otimes_k B \xrightarrow{(p,Id)} B \longrightarrow 0$$

splits topologically. In particular, we obtain a topological isomorphism

$$\Delta \otimes_A B \;=\; \Delta_p \ .$$

Now, let us consider the following exact sequence of locally convex A-modules

$$0 \longrightarrow \Delta^2 \longrightarrow \Delta \xrightarrow{\pi} \mathfrak{D}_{A/k} \longrightarrow 0 \ ,$$

where π is an open map. Applying the functor $\otimes_A B$ we obtain an exact sequence

$$\Delta^2 \otimes_A B \longrightarrow \Delta \otimes_A B = \Delta_p \xrightarrow{\pi\otimes 1} \mathfrak{D}_{A/k} \otimes_A B \longrightarrow 0 \ ,$$

where $\pi \otimes 1$ is an open map (6.3.a). Taking generators, it is easy to check that the image of $\Delta^2 \otimes_A B$ in $A \otimes_k B$ is Δ_p^2. Therefore, we have a topological isomorphism

$$\Delta_p/\Delta_p^2 \;=\; \mathfrak{D}_{A/k} \otimes_A B$$

and by completion we conclude that

$$\mathcal{D}_p/\overline{\mathcal{D}_p^2} \;=\; \Omega_{A/k}\widehat{\otimes}_A B \ .$$

□

Proposition 10.8. *The module of relative differentials is stable under base change: If $k \to A$ and $k \to K$ are morphisms of differentiable algebras, then*

$$\Omega_{(A\widehat{\otimes}_k K)/K} \ = \ \Omega_{A/k}\widehat{\otimes}_k K \ .$$

In particular, if $\Omega_{A/k}$ is a finitely generated projective A-module, then

$$\Omega_{(A\widehat{\otimes}_k K)/K} \ = \ \Omega_{A/k} \otimes_k K \ .$$

Proof. Let us consider the morphism $p\colon A \to A\widehat{\otimes}_k K$, $p(a) = a \otimes 1$. By 10.7, we have

$$\Omega_{A/k}\widehat{\otimes}_A (A\widehat{\otimes}_k K) \ = \ \Omega_p(A/k) \ .$$

On the other hand, $\mathcal{D}_p$ is the kernel of the natural morphism

$$\begin{array}{ccc} A\widehat{\otimes}_k A\widehat{\otimes}_k K & \xrightarrow{(p,Id)} & A\widehat{\otimes}_k K \\ \| & & \\ (A\widehat{\otimes}_k K)\widehat{\otimes}_K (A\widehat{\otimes}_k K) & & \end{array}$$

so that $\mathcal{D}_p = \mathcal{D}_{(A\widehat{\otimes}_k K)/K}$ and we conclude that $\Omega_p(A/k) = \Omega_{(A\widehat{\otimes}_k K)/K}$.

□

10.2 Exact Sequences of Differentials

First sequence of differentials. *Let $k \to A \to B$ be morphisms of differentiable algebras. We have a cokernel of Fréchet B-modules*

$$\Omega_{A/k}\widehat{\otimes}_A B \ \longrightarrow \ \Omega_{B/k} \ \longrightarrow \ \Omega_{B/A} \ \longrightarrow \ 0 \ .$$

If $\Omega_{A/k}\widehat{\otimes}_A B \to \Omega_{B/k}$ admits a continuous retraction, then this sequence is exact and it splits topologically.

Proof. The natural continuous epimorphism $B \otimes_k B \to B \otimes_A B$ is open, and the inverse image of $\Delta_{B/A}$ is just $\Delta_{B/k}$, so that the natural continuous morphism $\Delta_{B/k} \to \Delta_{B/A}$ is an open epimorphism; hence so is the induced morphism

$$\mathfrak{D}_{B/k} = \Delta_{B/k}/\Delta^2_{B/k} \ \longrightarrow \ \Delta_{B/A}/\Delta^2_{B/A} = \mathfrak{D}_{B/A} \ .$$

Therefore, the standard exact sequence of differentials

$$\mathfrak{D}_{A/k} \otimes_A B \ \longrightarrow \ \mathfrak{D}_{B/k} \ \longrightarrow \ \mathfrak{D}_{B/A} \ \longrightarrow \ 0$$

is a cokernel of locally convex B-modules. Applying completion (see 6.1.b and 10.3) we obtain the desired sequence.

Finally, if the morphism $\Omega_{A/k}\widehat{\otimes}_A B \to \Omega_{B/k}$ admits a continuous retraction, then it is injective and $\Omega_{A/k}\widehat{\otimes}_A B$ has the induced topology. Hence its image is closed and we conclude that

$$\Omega_{B/k} \ = \ (\Omega_{A/k}\widehat{\otimes}_A B) \oplus \Omega_{B/A} \ .$$

□

Corollary 10.9. $\Omega_{(A\widehat{\otimes}_k B)/k} = (\Omega_{A/k}\widehat{\otimes}_k B) \oplus (A\widehat{\otimes}_k \Omega_{B/k})$.

Proof. Let us consider the sequences of differentials corresponding to the canonical morphisms $A \to A\widehat{\otimes}_k B$ and $B \to A\widehat{\otimes}_k B$:

$$\begin{array}{ccccccccc} & & \Omega_{A/k}\widehat{\otimes}_A(A\widehat{\otimes}_k B) & \longrightarrow & \Omega_{(A\widehat{\otimes}_k B)/k} & \longrightarrow & \Omega_{B/k}\widehat{\otimes}_B(A\widehat{\otimes}_k B) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longleftarrow & \Omega_{A/k}\widehat{\otimes}_A(A\widehat{\otimes}_k B) & \longleftarrow & \Omega_{(A\widehat{\otimes}_k B)/k} & \longleftarrow & \Omega_{B/k}\widehat{\otimes}_B(A\widehat{\otimes}_k B) & & \end{array}$$

So we see that the continuous morphism $\Omega_{A/k}\widehat{\otimes}_A(A\widehat{\otimes}_k B) \to \Omega_{(A\widehat{\otimes}_k B)/k}$ admits a continuous retraction $\Omega_{(A\widehat{\otimes}_k B)/k} \to \Omega_{A/k}\widehat{\otimes}_A(A\widehat{\otimes}_k B)$; hence these sequences split topologically:

$$\Omega_{(A\widehat{\otimes}_k B)/k} = \left(\Omega_{A/k}\widehat{\otimes}_A(A\widehat{\otimes}_k B)\right) \oplus \left(\Omega_{B/k}\widehat{\otimes}_B(A\widehat{\otimes}_k B)\right) .$$

□

Second sequence of differentials. *Let $k \to A$ be a morphism of differentiable algebras. If $\mathfrak{a}$ is a closed ideal of A and we put $B = A/\mathfrak{a}$, then we have a cokernel of Fréchet B-modules*

$$\mathfrak{a}/\overline{\mathfrak{a}^2} \xrightarrow{\ \mathrm{d}\ } \Omega_{A/k}\widehat{\otimes}_A B \longrightarrow \Omega_{B/k} \longrightarrow 0 ,$$

where $\mathrm{d}[a] = (\mathrm{d}a)\otimes 1$. *If the morphism* $\mathrm{d}\colon \mathfrak{a}/\overline{\mathfrak{a}^2} \to \Omega_{A/k}\widehat{\otimes}_A B$ *admits a continuous retraction, then this sequence is exact and it splits topologically.*

Proof. The natural morphism $\Delta_{A/k} \oplus A = A \otimes_k A \to B \otimes_k B = \Delta_{B/k} \oplus B$ is an open epimorphism, hence so is the morphism $\Delta_{A/k} \to \Delta_{B/k}$, and the morphism $\mathfrak{D}_{A/k} = \Delta_{A/k}/\Delta^2_{A/k} \to \Delta_{B/k}/\Delta^2_{B/k} = \mathfrak{D}_{B/k}$. Therefore, the standard exact sequence

$$\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\ \mathrm{d}\ } \mathfrak{D}_{A/k} \otimes_A B \longrightarrow \mathfrak{D}_{B/k} \longrightarrow 0$$

is a cokernel of locally convex B-modules. Applying completion (6.1.b and 10.3) we obtain the desired second sequence of differentials.

If the morphism $\mathfrak{a}/\overline{\mathfrak{a}^2} \to \Omega_{A/k}\widehat{\otimes}_A B$ admits a continuous retraction, the argument given in the proof of the first sequence of differentials shows that the second sequence of differentials splits topologically.

□

Lemma 10.10. *The diagonal ideal $\mathcal{D}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ is generated by the increments of the cartesian coordinates $\Delta x_1 = x_1 - y_1, \ldots, \Delta x_n = x_n - y_n$.*

Proof. $\mathcal{D}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ is the ideal of the the diagonal embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$. In other words, $\mathcal{D}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ is the ideal of all functions in $\mathcal{C}^\infty(\mathbb{R}^n)\widehat{\otimes}_{\mathbb{R}}\mathcal{C}^\infty(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ vanishing on the diagonal submanifold $x_1 = y_1, \ldots, x_n = y_n$. By 2.7, the elements $\Delta x_i := x_i \otimes 1 - 1 \otimes y_i = x_i - y_i$ generate $\mathcal{D}_{\mathcal{C}^\infty(\mathbb{R}^n)}$.

□

Proposition 10.11. *The module $\Omega_{\mathcal{C}^\infty(\mathbb{R}^n)}$ is free and $\{\mathrm{d}x_1, \dots \mathrm{d}x_n\}$ is a basis:*

$$\Omega_{\mathcal{C}^\infty(\mathbb{R}^n)} \;=\; \mathcal{C}^\infty(\mathbb{R}^n)\mathrm{d}x_1 \oplus \cdots \oplus \mathcal{C}^\infty(\mathbb{R}^n)\mathrm{d}x_n \ .$$

Therefore, if U is an open subset of $\mathbb{R}^n$, then $\{\mathrm{d}x_1, \dots \mathrm{d}x_n\}$ is a basis of the free $\mathcal{C}^\infty(U)$-module $\Omega_{\mathcal{C}^\infty(U)}$.

Proof. By 10.10, we have that $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ generate $\Omega_{\mathcal{C}^\infty(\mathbb{R}^n)}$. On the other hand, given a relation $f_1\mathrm{d}x_1 + \cdots + f_n\mathrm{d}x_n = 0$ with coefficients in $\mathcal{C}^\infty(\mathbb{R}^n)$, applying 10.4 to the continuous derivation $\partial/\partial x_i \colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$, we obtain

$$0 = (f_1\mathrm{d}x_1 + \cdots + f_n\mathrm{d}x_n)(\partial/\partial x_i) = f_i \ .$$

Finally, the case of an open subset of $\mathbb{R}^n$ follows directly from 10.5.

□

Corollary 10.12. *Let $k \to A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ be a morphism of differentiable algebras. The A-module of relative differentials $\Omega_{A/k}$ is generated by* $\mathrm{d}x_1, \dots, \mathrm{d}x_n$.

Proof. By 10.11 and the second sequence of differentials applied to the ideal $\mathfrak{a}$ of $\mathcal{C}^\infty(\mathbb{R}^n)$, we obtain that Ω_A is generated by $\mathrm{d}x_1, \dots, \mathrm{d}x_n$. Now, the result follows from the first sequence of differentials applied to $\mathbb{R} \to k \to A$.

□

Corollary 10.13. *Let A be a differentiable algebra and let $x \in X = \mathrm{Spec}_r\, A$. Then*

$$\Omega_A \otimes_A (A/\mathfrak{m}_x) \;=\; \mathfrak{m}_x/\mathfrak{m}_x^2 \;=\; T_x^*X \ .$$

Proof. By 10.6 and 10.7 we have

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \;=\; \Omega_x(A/\mathbb{R}) \;=\; \Omega_A\widehat{\otimes}_A(A/\mathfrak{m}_x) \;=\; \Omega_A/\,\overline{\mathfrak{m}_x\Omega_A} \ .$$

Since Ω_A is a finitely generated A-module, there exists a continuous epimorphism $A^m \to \Omega_A$, which is an open map since both modules are Fréchet. Then

$$\mathbb{R}^m = A^m \otimes_A (A/\mathfrak{m}_x) \;\longrightarrow\; \Omega_A \otimes_A (A/\mathfrak{m}_x) \;=\; \Omega_A/\mathfrak{m}_x\Omega_A$$

is an open epimorphism, hence $\Omega_A/\mathfrak{m}_x\Omega_A$ is separated, i.e., $\mathfrak{m}_x\Omega_A = \overline{\mathfrak{m}_x\Omega_A}$.

Finally, $\mathfrak{m}_x/\mathfrak{m}_x^2 = T_x^*X$ by 5.12.

□

In general these sequences of differentials are not exact. Let us show an example: Let $h(x)$ be a differentiable function on $\mathbb{R}$ with null Taylor series at the point $x = 0$ and no other zero, and let $\psi(x)$ be a primitive of $h(x)$. Let us consider the first sequence of differentials corresponding to the morphism $A = \mathcal{C}^\infty(\mathbb{R}) \to B = \mathcal{C}^\infty(\mathbb{R})$, $f(y) \mapsto f(\psi(x))$, induced by the differentiable map $\psi \colon \mathbb{R} \to \mathbb{R}$, $y = \psi(x)$:

$$\begin{array}{ccccccc} \Omega_A\widehat{\otimes}_A B & \xrightarrow{\psi^*} & \Omega_B & \longrightarrow & \Omega_{B/A} & \longrightarrow & 0 \\ \| & & \| & & & & \\ B\,\mathrm{d}y = (A\,\mathrm{d}y)\widehat{\otimes}_A B & & B\,\mathrm{d}x & & & & \end{array}$$

where $\psi^*(\mathrm{d}y) = \mathrm{d}(\psi(x)) = h(x)\mathrm{d}x$. Hence the image $h(x)B\mathrm{d}x$ of ψ^* is not closed in $B\,\mathrm{d}x = \Omega_B$ and, in particular, it does not coincide with the kernel of $\Omega_B \to \Omega_{B/A}$.

10.3 Sheaf of Relative Differentials

Proposition 10.14. *Let $\varphi\colon X = \operatorname{Spec}_r A \to \operatorname{Spec}_r k = S$ be a morphism of affine differentiable spaces. Let us consider open subsets $V \subseteq S$ and $U \subseteq \varphi^{-1}V$. We have*

$$\Omega_{A_U/k_V} \;=\; \Omega_{A_U/k}\ .$$

Proof. Note that

$$A_U \widehat{\otimes}_k k_V \overset{8.11}{=} (A_U)_V \;=\; A_U\ .$$

Since the module of relative differentials is stable under base change, we have

$$\Omega_{A_U/k_V} \;=\; \Omega_{(A_U\widehat{\otimes}_k k_V)/k_V} \overset{10.8}{=} \Omega_{A_U/k}\widehat{\otimes}_k k_V \overset{8.11}{=} (\Omega_{A_U/k})_V \;=\; \Omega_{A_U/k}\ .$$

□

Definition. Let $\varphi\colon X = \operatorname{Spec}_r A \to \operatorname{Spec}_r k = S$ be a morphism of affine differentiable spaces. The sheaf of relative differentials of X over S is defined to be $\Omega_{X/S} := \widetilde{\Omega}_{A/k}$. By the Localization theorem for Fréchet modules, we have

$$\Omega_{X/S}(U) \;=\; (\Omega_{A/k})_U \overset{10.5}{=} \Omega_{A_U/k}$$

for any open subset $U \subseteq X$.

By 10.14, for any open subsets $V \subseteq S$ and $U \subseteq \varphi^{-1}V$ we have

$$(\Omega_{X/S})_{|U} \;=\; \Omega_{U/S} \;=\; \Omega_{U/V}\ .$$

This fact let us define the sheaf $\Omega_{X/S}$ in the non-affine case by "recollement": Let $\varphi\colon X \to S$ be a morphism of differentiable spaces. The **sheaf of relative differentials** of X over S is defined to be the unique $\mathcal{O}_X$-module $\Omega_{X/S}$ such that

$$(\Omega_{X/S})_{|U} \;=\; \Omega_{U/V}$$

for any affine open subsets $V \subseteq S$ and $U \subseteq \varphi^{-1}V$. When $S = \operatorname{Spec}_r \mathbb{R}$, we denote it by Ω_X.

According to 10.11 and its corollaries, we have the following facts:

1. $\Omega_{X/S}$ is a locally finitely generated $\mathcal{O}_X$-module.
2. $\Omega_{\mathbb{R}^n}$ is a free $\mathcal{O}_{\mathbb{R}^n}$-module. Moreover, $\Omega_{\mathbb{R}^n}$ coincides with the sheaf of differentiable 1-forms on $\mathbb{R}^n$ (this fact is easily generalized to smooth manifolds).
3. If $x \in X$, then $T_x^*X \;=\; \mathfrak{m}_x/\mathfrak{m}_x^2 \;=\; \Omega_{X,x}/\mathfrak{m}_x\Omega_{X,x}$.

10.4 Smooth Morphisms

Definition. A morphism of differentiable spaces $\varphi\colon X \to S$ is said to be **smooth** at a point $x \in X$ (or that X is smooth over S at x) if there exists an open neighbourhood U of x in X such that $\varphi(U)$ is open in S and there exists a S-isomorphism $U \simeq V \times \varphi(U)$, where V is an open subspace of some $\mathbb{R}^n$. In such a case we say that n is the **relative dimension** of φ at x.

We say that φ is a smooth morphism when so it is at any point of X.

The following properties of smooth morphisms follow directly from the definition:

1. If $\phi\colon Y \to X$ and $\varphi\colon X \to S$ are smooth morphisms, then so is the composition $\varphi\phi\colon Y \to S$.
2. The concept of smooth morphism is stable under base change: If a morphism $X \to S$ is smooth, then so is $X \times_S T \to T$ for any morphism $T \to S$.
3. The concept of smooth morphism $\varphi\colon X \to S$ is local in S: If $\{U_i\}$ is an open cover of S and any morphism $\varphi|_{\varphi^{-1}(U_i)}\colon \varphi^{-1}(U_i) \to U_i$ is smooth, then φ is a smooth morphism.
4. The concept of smooth morphism $\varphi\colon X \to S$ is local in X: If $\{V_j\}$ is an open cover of X and any morphism $\varphi|_{V_j}\colon V_j \to S$ is smooth, then φ is a smooth morphism.
5. Open embeddings (in general local isomorphisms) are smooth morphisms.
6. Smooth differentiable spaces over $\mathrm{Spec}_r\,\mathbb{R}$ are just smooth manifolds. Hence, the fibres X_s of any smooth morphism $X \to S$ are smooth manifolds. If $\mathcal{V}$ is a smooth manifold and S is any differentiable space, then the second projection $\mathcal{V} \times S \to S$ is a smooth morphism.
7. A differentiable map $\mathcal{V} \to \mathcal{W}$ between smooth manifolds is a smooth morphism at a point $p \in \mathcal{V}$ if and only if it is a submersion at p.

Theorem 10.15. *Let X be a differentiable space whose underlying topological space is a topological manifold of constant dimension n. Then X is a smooth manifold if and only if the sheaf of differentials Ω_X is a locally free $\mathcal{O}_X$-module of rank n.*

Proof. If X is a smooth manifold and U is an open subspace of X which is isomorphic to an open subspace of $\mathbb{R}^r$, then $r = n$ by the invariance of domain ([68] 4.7.16). By 10.11 we have $\Omega_X|_U = \Omega_U = \mathcal{O}_U \mathrm{d}x_1 \oplus \ldots \oplus \mathcal{O}_U \mathrm{d}x_n$, and we conclude that Ω_X is a locally free $\mathcal{O}_X$-module of rank n.

Conversely, let us assume that Ω_X is a locally free $\mathcal{O}_X$-module of rank n. Let $p \in X$ and let $x_1, \ldots, x_n$ be differentiable functions on an open neighbourhood U of x such that $\{\mathrm{d}_p x_1, \ldots, \mathrm{d}_p x_n\}$ is a basis of

$$T_p^* X \;=\; \mathfrak{m}_p/\mathfrak{m}_p^2 \;=\; \Omega_{X,p}/\mathfrak{m}_p \Omega_{X,p} \;\simeq\; (\mathcal{O}_{X,p}/\mathfrak{m}_p)^n \;.$$

By 5.21, the morphism

$$(x_1, \ldots, x_n)\colon U \longrightarrow \mathbb{R}^n$$

is a local embedding at x. Therefore, restricting U if necessary, we may assume that it induces an isomorphism of U with a closed differentiable subspace Y of an open subspace of $\mathbb{R}^n$. Since U is a topological manifold of dimension n, then Y is open in $\mathbb{R}^n$ by the invariance of domain; hence X is a smooth manifold of dimension n.

□

Proposition 10.16. *If $\varphi\colon X \to S$ is a smooth morphism of constant relative dimension n, then $\Omega_{X/S}$ is a locally free $\mathcal{O}_X$-module of rank n.*

Proof. Since the question is local and φ is a smooth morphism, we may assume that $S = \operatorname{Spec}_r k$ is affine, $X = S \times \mathbb{R}^n$ and that $\varphi\colon X \to S$ is the natural projection. Then we have that $X = \operatorname{Spec}_r A$, where $A := k \,\widehat{\otimes}_{\mathbb{R}}\, \mathcal{C}^\infty(\mathbb{R}^n)$. The first sequence of differentials

$$\Omega_k \widehat{\otimes}_{\mathbb{R}} \mathcal{C}^\infty(\mathbb{R}^n) = \Omega_k \widehat{\otimes}_k A \longrightarrow \Omega_A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

and the equalities (see 10.9 and 10.11)

$$\begin{aligned} \Omega_A &= (\Omega_k \widehat{\otimes}_{\mathbb{R}} \mathcal{C}^\infty(\mathbb{R}^n)) \oplus (\Omega_{\mathcal{C}^\infty(\mathbb{R}^n)} \widehat{\otimes}_{\mathbb{R}} k) \\ &= (\Omega_k \widehat{\otimes}_{\mathbb{R}} \mathcal{C}^\infty(\mathbb{R}^n)) \oplus (A\mathrm{d}x_1 \oplus \cdots \oplus A\mathrm{d}x_n) \end{aligned}$$

imply that $\Omega_{A/k} = A\mathrm{d}x_1 \oplus \cdots \oplus A\mathrm{d}x_n$, hence $\Omega_{X/S}$ is a locally free $\mathcal{O}_X$-module of rank n.

□

Theorem 10.17. *Let $\varphi\colon X \to S$ be a morphism of differentiable spaces whose topological fibres are topological manifolds of constant dimension n. Let us assume that S is reduced. Then φ is a smooth morphism if and only if φ is an open map and $\Omega_{X/S}$ is a locally free $\mathcal{O}_X$-module of rank n.*

Proof. ($\Rightarrow$). It is clear that any smooth morphism is an open map and, by 10.16, $\Omega_{X/S}$ is locally free of rank n.

($\Leftarrow$). Since the question is local and φ is an open map, we may assume that φ is a surjective morphism of affine differentiable spaces

$$\varphi\colon X = \operatorname{Spec}_r A \longrightarrow \operatorname{Spec}_r k = S$$

and that $\Omega_{A/k} = A\mathrm{d}x_1 \oplus \cdots \oplus A\mathrm{d}x_n$ is a free A-module.

Given a point $s \in S$, the fibre

$$Y := \varphi^{-1}(s) \;=\; \operatorname{Spec}_r(A/\overline{\mathfrak{m}_s A})$$

is a smooth manifold. In fact, since $A/\overline{\mathfrak{m}_s A} = A \widehat{\otimes}_k (k/\mathfrak{m}_s)$ and the differentials are stable under base change (10.8), we have that

$$\Omega_{(A/\overline{\mathfrak{m}_s A})/\mathbb{R}} \;=\; \Omega_{A/k} \widehat{\otimes}_k (k/\mathfrak{m}_s) \;=\; (A/\overline{\mathfrak{m}_s A})\mathrm{d}x_1 \oplus \cdots \oplus (A/\overline{\mathfrak{m}_s A})\mathrm{d}x_n$$

is a free module of rank n. By 10.15, we conclude that $Y = \varphi^{-1}(s)$ is a smooth manifold of dimension n. Moreover, the proof of 10.15 shows that $\phi = (x_1, \dots, x_n)\colon Y \to \mathbb{R}^n$ is a local diffeomorphism.

Now, given a point $x \in Y$, let us show that the following sequence of real vector spaces is exact:

$$0 \longrightarrow T_xY \longrightarrow T_xX \xrightarrow{\varphi_*} T_sS \qquad (*)$$

Applying the functor $\widehat{\otimes}_A(A/\mathfrak{m}_{X,x})$ to the first sequence of differentials

$$\Omega_k\widehat{\otimes}_k A \longrightarrow \Omega_A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

we obtain the exact sequence

$$\mathfrak{m}_s/\mathfrak{m}_s^2 \longrightarrow \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \longrightarrow \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2 \longrightarrow 0\ ,$$

or equivalently (5.12),

$$T_s^*S \xrightarrow{\varphi^*} T_xX^* \longrightarrow T_x^*Y \longrightarrow 0\ ,$$

and the dual sequence is just the desired exact sequence.

By 5.21, the morphism $(\varphi, \phi) = (\varphi, x_1, \dots, x_n)\colon X \to S \times \mathbb{R}^n$ is a local embedding, because

$$(\varphi_*, \phi_*)\colon T_xX \longrightarrow T_sS \oplus T_{\varphi(x)}\mathbb{R}^n$$

is injective (recall the exact sequence $(*)$ and that $\phi\colon Y \to \mathbb{R}^n$ is a local diffeomorphism). Therefore, we may assume that X is a closed differentiable subspace of $S \times \mathbb{R}^n$. Moreover, since the fibres of both spaces over S are topological manifolds of dimension n, we obtain easily (by the invariance of domain) that the closed embedding $X \to S \times \mathbb{R}^n$ is bijective. Finally, $S \times \mathbb{R}^n$ being reduced, we conclude that $X = S \times \mathbb{R}^n$.

□

Proposition 10.18. *Let $\varphi\colon X \to S$ be a finite flat morphism of differentiable spaces. The following conditions are equivalent:*

1. *φ is a smooth morphism.*
2. *$\Omega_{X/S} = 0$.*
3. *φ is a local isomorphism at any point $x \in X$.*

(See also 9.16.)

Proof. (1) ⇒ (2). It is a particular case of 10.16, since any finite smooth morphism has relative dimension 0.

(2) ⇒ (3). Since the differentials are stable under base change, for any point $s \in S$, we have

$$\Omega_{\varphi^{-1}(s)} = \Omega_{X/S} \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathfrak{m}_s = 0$$

and by 10.15 we obtain that $\varphi^{-1}(s)$ is a smooth manifold of dimension 0. In other words, $\varphi^{-1}(s)$ is a finite reduced differentiable space, hence the index of ramification of φ at any point $x \in \varphi^{-1}(s)$ is 1. By 9.16 we conclude that φ is a local isomorphism at any point $x \in X$.

Finally, the equivalence (1) $\Leftrightarrow$ (3) is obvious.

□

Formally Smooth Spaces

Definition. A differentiable space X is said to be **formally smooth** if the sheaf of differentials Ω_X is a locally free $\mathcal{O}_X$-module (of finite rank by 10.12).

Smooth manifolds are formally smooth by 10.11.

Given a closed subset $Y \subseteq \mathbb{R}^n$, let us denote by $\mathbf{W}_Y$ the Whitney subspace of Y in $\mathbb{R}^n$. Recall (5.10) that $\mathbf{W}_Y$ is the differentiable subspace of $\mathbb{R}^n$ defined by the Whitney ideal

$$W_Y = \bigcap_{y,r} \mathfrak{m}_y^{r+1} \qquad (y \in Y,\ r \in \mathbb{N})$$

i.e., W_Y is the ideal of all differentiable functions on $\mathbb{R}^n$ whose Taylor expansion at any point $y \in Y$ is null.

Lemma 10.19. *Let Y be a closed set in $\mathbb{R}^n$. The Whitney subspace $\mathbf{W}_Y$ is formally smooth.*

Proof. $\mathbf{W}_Y$ is defined by the differentiable algebra $A = \mathcal{C}^\infty(\mathbb{R}^n)/W_Y$. Let us consider the second sequence of differentials

$$W_Y/\overline{W_Y^2} \longrightarrow \Omega_{\mathcal{C}(\mathbb{R}^n)} \widehat{\otimes}_{\mathcal{C}(\mathbb{R}^n)} A \longrightarrow \Omega_A \longrightarrow 0 .$$

By Whitney's spectral theorem, we have $\overline{W_Y^2} = W_Y$, hence

$$\Omega_A = \Omega_{\mathcal{C}(\mathbb{R}^n)} \widehat{\otimes}_{\mathcal{C}(\mathbb{R}^n)} A \overset{10.11}{=} A\mathrm{d}x_1 \oplus \cdots \oplus A\mathrm{d}x_n ,$$

that is to say, $\Omega_{\mathbf{W}_Y}$ is a free $\mathcal{O}_{\mathbf{W}_Y}$-module.

□

Theorem 10.20. *A differentiable space X is formally smooth if and only if each point $p \in X$ has an open neighbourhood isomorphic to the Whitney subspace of a closed set of some $\mathbb{R}^n$.*

Proof. If X is formally smooth, then we may assume that Ω_X is a free $\mathcal{O}_X$-module:

$$\Omega_X = \mathcal{O}_X \mathrm{d}x_1 \oplus \ldots \oplus \mathcal{O}_X \mathrm{d}x_n .$$

By 10.13, $\{\mathrm{d}_p x_1, \ldots, \mathrm{d}_p x_n\}$ is a basis of the cotangent space $T_p^* X$ and, according to 5.21, the differentiable functions $x_1, \ldots, x_n$ define a local embedding

$X \hookrightarrow \mathbb{R}^n$. That is to say, we may assume that X is a closed differentiable subspace of $\mathbb{R}^n$ and that $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ is a basis of Ω_X. Let us write $X = \operatorname{Spec}_r A$ where $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$. Since $\Omega_A = A\mathrm{d}x_1 \oplus \cdots \oplus A\mathrm{d}x_n$, the morphism π in the second sequence of differentials

$$\mathfrak{a}/\overline{\mathfrak{a}^2} \xrightarrow{\mathrm{d}} \Omega_{\mathcal{C}^\infty(\mathbb{R}^n)} \widehat{\otimes}_{\mathcal{C}^\infty(\mathbb{R}^n)} A \overset{10.11}{=} A\mathrm{d}x_1 \oplus \cdots \oplus A\mathrm{d}x_n \xrightarrow{\pi} \Omega_A \longrightarrow 0$$

is an isomorphism. Therefore, the morphism

$$\mathfrak{a}/\overline{\mathfrak{a}^2} \xrightarrow{\mathrm{d}} A\mathrm{d}x_1 \oplus \cdots \oplus A\mathrm{d}x_n \ , \qquad [f] \mapsto \sum_i [\partial f/\partial x_i]\mathrm{d}x_i$$

is null. In other words, $\frac{\partial}{\partial x_i}(\mathfrak{a}) \subseteq \mathfrak{a}$ for any index i. This fact implies that the Taylor expansion of any function $f \in \mathfrak{a}$ at any point $x \in (\mathfrak{a})_0 = X$ is null. By Whitney's spectral theorem we conclude that $\mathfrak{a} = W_X$, hence X is a Whitney subspace.

Conversely, such condition is sufficient because the concept of formally smooth differentiable space is local, and $\mathbf{W}_X$ is formally smooth by 10.19.

□

Note. The proof of theorem 10.20 shows in fact that a differentiable algebra A is the Whitney algebra of a closed set in $\mathbb{R}^n$ if and only if the module of differentials is a free module $\Omega_A = A\mathrm{d}x_1 \oplus \ldots \oplus A\mathrm{d}x_n$, where $x_1, \ldots, x_n$ generate A as a $\mathcal{C}^\infty$-ring (when A has compact spectrum, by 5.25 and 5.26 this condition states that $\mathbb{R}[x_1, \ldots, x_n]$ is a dense subalgebra). Within a certain class of Fréchet algebras with compact spectrum, Whitney algebras of compact sets $K \subset \mathbb{R}^n$ were characterized in [35] by the following properties:

1. There exist n functions $x_1, \ldots, x_n \in A$ such that $\mathbb{R}[x_1, \ldots, x_n]$ is dense in A.
2. The diagonal ideal $\mathcal{D}_A$ is a finitely generated ideal and any power $\mathcal{D}_A^r$ is closed in $A \widehat{\otimes}_{\mathbb{R}} A$.
3. Ω_A is a free module of rank n and the graded ring $\bigoplus_r (\mathcal{D}_A^r/\mathcal{D}_A^{r+1})$ is the symmetric algebra of Ω_A.

and, within a more restrictive class of Fréchet algebras, Whitney algebras of compact sets in smooth manifolds were characterized in [41] by the following properties:

1. The diagonal ideal $\mathcal{D}_A$ is a finitely generated ideal and any power $\mathcal{D}_A^r$ is closed in $A \widehat{\otimes}_{\mathbb{R}} A$.
2. Ω_A is a projective A-module and the graded ring $\bigoplus_r (\mathcal{D}_A^r/\mathcal{D}_A^{r+1})$ is the symmetric algebra of Ω_A.

For analogous characterizations of algebras $\mathcal{C}^\infty(\mathcal{V})$ of differentiable functions on smooth manifolds we refer the reader to [38] and [10].

11 Quotients by Compact Lie Groups

Let us consider a differentiable action of a compact Lie group G on a smooth manifold X. In general, the topological quotient X/G does not admit a smooth structure. For example, if $G = \{\pm 1\}$ acts by multiplication on $\mathbb{R}^3$, then the quotient $\mathbb{R}^3/G$ is not even a topological manifold. More generally, the quotient of $\mathbb{R}^n$ by a non-trivial linear action of G is not a smooth manifold. This kind of example shows that the category of smooth manifolds is too restrictive for an adequate analysis of quotients.

The main purpose of this chapter is to state the existence of quotients, with respect to the actions of compact Lie groups, in the category of differentiable spaces. Given a differentiable action of a compact Lie group G on a differentiable space X, let us consider the topological quotient $\pi : X \to X/G$ endowed with the following sheaf of rings $\mathcal{O}_{X/G}$:

$$\mathcal{O}_{X/G}(U) \ := \ \mathcal{O}_X(\pi^{-1}U)^G$$

for any open subset U of X/G, where $\mathcal{O}_X(\pi^{-1}U)^G$ denotes the algebra of all G-invariant differentiable functions on $\pi^{-1}U$. The morphism of ringed spaces $\pi : (X, \mathcal{O}_X) \longrightarrow (X/G, \mathcal{O}_{X/G})$ is called the *geometric quotient* of X by G. Using Mostow's equivariant embedding theorem and Schwartz's theorem on differential invariants, we shall prove that the geometric quotient $(X/G, \mathcal{O}_{X/G})$ is a differentiable space.

In the case of X being a smooth manifold, the quotient X/G is not a smooth manifold in general. However, the differentiable structure of X/G is not very far from a smooth one, since it admits a locally finite stratification by locally closed smooth submanifolds. Moreover, the set of all non-singular points of X/G is a dense open subset (we say that a point of X/G is non-singular when it has an open neighbourhood which is a smooth manifold).

11.1 Godement's Theorem

Classical classification problems may be viewed as the determination of the structure of the quotient $\mathcal{V}/R$ of a certain family of objects $\mathcal{V}$ under an equivalence relation $R \subseteq \mathcal{V} \times \mathcal{V}$. In most cases $\mathcal{V}$ is a smooth manifold and it is natural to study whether the quotient set $\mathcal{V}/R$ admits a natural structure of smooth manifold such that the canonical projection $\pi \colon \mathcal{V} \to \mathcal{V}/R$ is a differentiable map.

J.A. Navarro González and J.B. Sancho de Salas: LNM 1824, pp. 127–150, 2003.

Definition. Let $R \subseteq \mathcal{V} \times \mathcal{V}$ be an equivalence relation on a smooth manifold $\mathcal{V}$ and let $\pi \colon \mathcal{V} \to \mathcal{V}/R$ be the quotient map. We shall consider the quotient topology on $\mathcal{V}/R$. This space is endowed with the following sheaf of rings:

$$\mathcal{O}_{\mathcal{V}/R}(U) \;=\; \begin{bmatrix} \text{Continuous functions } f\colon U \to \mathbb{R} \\ \text{such that } f \circ \pi \in \mathcal{C}^\infty(\pi^{-1}U) \end{bmatrix} \,.$$

If the pair $(\mathcal{V}/R, \mathcal{O}_{\mathcal{V}/R})$ is a smooth manifold, then we say that $(\mathcal{V}/R, \mathcal{O}_{\mathcal{V}/R})$ is the **quotient manifold** of $\mathcal{V}$ by R. Note that $\pi \colon \mathcal{V} \to \mathcal{V}/R$ is a differentiable map in this case.

The quotient manifold $\mathcal{V}/R$ has the adequate categorial property: If a differentiable map $\varphi \colon \mathcal{V} \to \mathcal{W}$ to a smooth manifold $\mathcal{W}$ is constant on the equivalence classes of R, then there exists a unique differentiable map $\psi \colon \mathcal{V}/R \to \mathcal{W}$ such that $\varphi = \psi \circ \pi$.

Definitions. An **action** of a group G on a set X is a map $\mu \colon G \times X \to X$ such that $1 \cdot x = x$ and $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for any $x \in X$, $g_1, g_2 \in G$, where we put $g \cdot x := \mu(g, x)$.

Let X and Y be sets endowed with an action of a group G. A map $\varphi \colon X \to Y$ is said to be **G-equivariant** when $\varphi(g \cdot x) = g \cdot \varphi(x)$ for any $x \in X$, $g \in G$.

Given an action $\mu \colon G \times X \to X$, the image R of $\varrho \colon G \times X \to X \times X$, $\varrho(g, x) = (g \cdot x, x)$, defines an equivalence relation on X:

$$x \equiv x' \;\Leftrightarrow\; x = g \cdot x' \text{ for some } g \in G \,.$$

In this case the quotient set X/R will be denoted by X/G.

The equivalence class $Gx = \{g \cdot x;\; g \in G\}$ of any point $x \in X$ is said to be the **orbit** of x. The **isotropy subgroup** of x is defined to be

$$I_x \;:=\; \{g \in G \colon\; g \cdot x = x\} \,.$$

Note that the natural map $G/I_x \to Gx$, $[g] \mapsto g \cdot x$ is bijective.

An action of a topological group G on a topological space X is said to be a **continuous action** if the map $\mu \colon G \times X \to X$ is continuous.

Analogously, an action of a Lie group G on a smooth manifold $\mathcal{V}$ is said to be a **differentiable action** if the map $\mu \colon G \times \mathcal{V} \to \mathcal{V}$ is differentiable. Let $\pi \colon \mathcal{V} \to \mathcal{V}/G$ be the quotient map. As in the general case of a quotient by an equivalence relation, we consider on $\mathcal{V}/G$ the quotient topology and the structural sheaf

$$\mathcal{O}_{\mathcal{V}/G}(U) \;=\; \begin{bmatrix} \text{Continuous functions } f\colon U \to \mathbb{R} \\ \text{such that } f \circ \pi \in \mathcal{C}^\infty(\pi^{-1}U) \end{bmatrix} \,.$$

If the pair $(\mathcal{V}/G, \mathcal{O}_{\mathcal{V}/G})$ is a smooth manifold, then we say that $(\mathcal{V}/G, \mathcal{O}_{\mathcal{V}/G})$ is the **quotient manifold** of $\mathcal{V}$ by G.

Proposition 11.1. *Let $R \subseteq \mathcal{V} \times \mathcal{V}$ be an equivalence relation on a smooth manifold $\mathcal{V}$. If $\varphi \colon \mathcal{V} \to \overline{\mathcal{V}}$ is a surjective submersion whose fibres coincide with the equivalence classes of R, then $\mathcal{V}/R$ is a smooth manifold and φ induces a diffeomorphism $\mathcal{V}/R = \overline{\mathcal{V}}$.*

Proof. Since any submersion is an open map, the natural bijection $\mathcal{V}/R = \overline{\mathcal{V}}$ is a homeomorphism. Finally, we have that $\mathcal{O}_{\mathcal{V}/R} = \mathcal{C}^\infty_{\overline{\mathcal{V}}}$ because a function f on $\overline{\mathcal{V}}$ is differentiable if and only if $f \circ \varphi$ is differentiable on $\mathcal{V}$ (1.22.e.1).

□

Godement's theorem. *Let R be an equivalence relation on a smooth manifold $\mathcal{V}$. The following conditions are equivalent:*

(a) *$\mathcal{V}/R$ is a smooth manifold and $\pi\colon \mathcal{V} \to \mathcal{V}/R$ is a submersion.*

(b) *R is a smooth submanifold of $\mathcal{V}\times\mathcal{V}$ and the second projection $\pi_2\colon R \to \mathcal{V}$ is a submersion (hence so is $\pi_1\colon R \to \mathcal{V}$).*

Moreover, in such a case, $\mathcal{V}/R$ is separated if and only if R is closed in $\mathcal{V}\times\mathcal{V}$.

Proof. $(a) \Rightarrow (b)$. If $\pi\colon \mathcal{V} \to \mathcal{V}/R$ is a surjective submersion, then 1.22.c let us conclude that $R = \{(x,y) \in \mathcal{V}\times\mathcal{V}\colon\ \pi(x) = \pi(y)\} = \mathcal{V}\times_{\mathcal{V}/R}\mathcal{V}$ is a smooth submanifold of $\mathcal{V}\times\mathcal{V}$ and that the second projection $R \to \mathcal{V}$ is a submersion.

$(b) \Rightarrow (a)$. Let us write $\mathcal{O} = \mathcal{O}_{\mathcal{V}/R}$. We have to prove that $(\mathcal{V}/R, \mathcal{O})$ is a smooth manifold and that $\pi\colon \mathcal{V} \to \mathcal{V}/R$ is a submersion.

1) π is an open map: If V is an open set in $\mathcal{V}$, then $\pi^{-1}(\pi V) = \pi_1(\pi_2^{-1}(V))$ is open, because $\pi_1\colon R \to \mathcal{V}$ is open, since so is any submersion.

2) To see that $(\mathcal{V}/R, \mathcal{O})$ is a smooth manifold, it is enough to prove that any point $p \in \mathcal{V}$ has an R-saturated (union of some equivalence classes) open neighbourhood U such that U/R is a smooth manifold, because U/R is open in $\mathcal{V}/R$ and $\mathcal{O}_{U/R} = \mathcal{O}|_{U/R}$ (since U is R-saturated). By the way, we shall prove that $U \to U/R$ is a submersion.

3) For any $p \in \mathcal{V}$, the equivalence class $[\,p\,]$ is a smooth submanifold of $\mathcal{V}$, because so are the fibres of any submersion and $[\,p\,]$ is just the fibre of $\pi_2\colon R \to \mathcal{V}$ over p.

4) p has an open neighbourhood V such that the quotient manifold of V by the induced equivalence relation exists and the quotient map is a submersion. In fact, if W is a smooth submanifold of $\mathcal{V}$ such that

$$T_p\mathcal{V} \ = \ T_p([\,p\,]) \oplus T_pW \ ,$$

then $R_W = \pi_2^{-1}(W)$ is a smooth submanifold of $\mathcal{V}\times W$ and we are going to prove that $\pi_1\colon R_W \to \mathcal{V}$ is a local diffeomorphism at (p,p). It is a submersion because the image of the tangent linear map contains

$$\begin{array}{ll} T_pW & \text{since } W \xrightarrow{Id\times Id} R_W \xrightarrow{\pi_1} \mathcal{V} \ \text{ is the identity on } W \ , \\ T_p([\,p\,]) & \text{since } [\,p\,] \xrightarrow{Id\times p} R_W \xrightarrow{\pi_1} \mathcal{V} \ \text{ is the identity on } [\,p\,] \ , \end{array}$$

and it is a local diffeomorphism because

$$\dim T_{(p,p)}(R_W) \ = \ \dim T_pW + \dim T_{(p,p)}([\,p\,]\times p) \ = \ \dim T_p\mathcal{V} \ .$$

Hence, we may assume the existence of open neighbourhoods U, V of p in $\mathcal{V}$ such that $\pi_1\colon (U \times U) \cap R_W \to V$ is a diffeomorphism. Let $x \mapsto (x, r(x))$ be the inverse map, where $r(x)$ is the *unique* point of W in $[x]$. The fibres of $r\colon V \to W$ are just the equivalence classes of the equivalence relation induced by R on V. Moreover, $V \cap W$ defines a local section of r passing through p. Replacing V by a smaller neighbourhood we may assume that r is a submersion and, replacing W by the open subset $r(V)$, we may assume that r is surjective. Hence $r\colon V \to W$ is the quotient manifold of V by the induced equivalence relation.

5) $\pi\colon \pi^{-1}(\pi V) \to \pi(V)$ is the quotient of $\pi^{-1}(\pi V)$ by R. In fact we have shown that $\pi(V)$ admits a structure of smooth manifold such that $\pi\colon V \to \pi(V)$ is a surjective submersion. Let us consider the following commutative square

$$\begin{array}{ccc} \pi_2^{-1}(V) & \xrightarrow{\pi_1} & \pi_1(\pi_2^{-1}V) = \pi^{-1}(\pi V) \\ {\scriptstyle \pi_2}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ V & \xrightarrow{\pi} & \pi(V) \end{array}$$

where the top, down and left arrows are surjective submersions. Hence so is the right arrow $\pi\colon \pi^{-1}(\pi V) \to \pi(V)$ by 1.22.b and 1.22.e.4. Since $U = \pi^{-1}(\pi V)$ is an R-saturated neighbourhood of p, step 2 let us conclude that $\mathcal{V}/R$ is a smooth manifold.

6) Finally, $\mathcal{V}/R$ is separated if and only if the diagonal Δ of $(\mathcal{V}/R) \times (\mathcal{V}/R)$ is closed. Now, $\pi\colon \mathcal{V} \to \mathcal{V}/R$ being a surjective open map, so is

$$\pi \times \pi\colon \mathcal{V} \times \mathcal{V} \longrightarrow (\mathcal{V}/R) \times (\mathcal{V}/R)\ ,$$

and we conclude that Δ is closed in $(\mathcal{V}/R)\times(\mathcal{V}/R)$ if and only if $R = (\pi\times\pi)^{-1}(\Delta)$ is closed in $\mathcal{V} \times \mathcal{V}$.

□

Corollary 11.2. *Let $\mu\colon G \times \mathcal{V} \to \mathcal{V}$ be a differentiable action of a Lie group G on a smooth manifold $\mathcal{V}$. The following conditions are equivalent:*

(a) *$\mathcal{V}/G$ is a smooth manifold and $\pi\colon \mathcal{V} \to \mathcal{V}/G$ is a submersion.*

(b) *The induced equivalence relation R is a smooth submanifold of $\mathcal{V} \times \mathcal{V}$.*

Proof. In fact, if R is a smooth submanifold, then $\pi_2\colon R \to \mathcal{V}$ is a submersion, since so is the composition $G\times\mathcal{V} \xrightarrow{\varrho} R \xrightarrow{\pi_2} \mathcal{V}$ and $\varrho(g, x) = (gx, x)$ is surjective. □

Corollary 11.3. *Let H be a closed Lie subgroup of a Lie group G and let us consider the right-action of H on G. Then G/H is a separated smooth manifold and $\pi\colon G \to G/H$ is a submersion.*

Therefore, if H is a normal subgroup, then G/H is a Lie group.

Proof. The image of the map $\varrho\colon H \times G \to G \times G$, $(h, g) \mapsto (gh, g)$, is a closed submanifold, because the diffeomorphism $G \times G \to G \times G$, $(a, b) \mapsto (b^{-1}a, b)$, transforms it into the closed submanifold $H \times G$.

□

Corollary 11.4. *Let $G \times \mathcal{V} \to \mathcal{V}$ be a differentiable action of a discrete group G on a smooth manifold $\mathcal{V}$. If each point $x \in \mathcal{V}$ has an open neighbourhood U such that $g_i U \cap g_j U = \emptyset$ whenever $i \neq j$, then $\mathcal{V}/G$ is a smooth manifold.*

Proof. In fact $\Gamma_g := \{(gx, x)\}_{x \in \mathcal{V}}$ is a smooth submanifold of $\mathcal{V} \times \mathcal{V}$ for any $g \in G$; hence so is $R = \bigcup_g \Gamma_g$, because any point $(gx, x) \in R$ has a neighbourhood $(gU) \times U$ which only intersects R at points of Γ_g.

□

Corollary 11.5. *Let $G \times \mathcal{V} \to \mathcal{V}$ be a differentiable free action of a finite group G on a separated smooth manifold $\mathcal{V}$. Then $\mathcal{V}/G$ is a separated smooth manifold.*

Proof. $\mathcal{V}/G$ is a smooth manifold by 11.4. Moreover, Γ_g is closed in $\mathcal{V} \times \mathcal{V}$ because $\mathcal{V}$ is assumed to be separated; hence $R = \Gamma_{g_1} \cup \ldots \cup \Gamma_{g_r}$ is closed and we conclude that $\mathcal{V}/G$ is separated.

□

Let us show that the separation hypothesis in 11.5 is necessary in order to conclude that $\mathcal{V}/G$ is a smooth manifold:

Example. Let $\widetilde{\mathbb{R}}$ be a real line where the origin $x = 0$ is split (see 1.3) in two points ± 0, and let us consider the natural multiplicative action of $G = \{\pm 1\}$ on $\widetilde{\mathbb{R}}$. It is a free action, but the topological quotient space $\widetilde{\mathbb{R}}/G \simeq [0, \infty)$ is not a topological manifold.

In the following example, we shall consider an elementary classification problem such that the corresponding quotient manifold is not Hausdorff, so explaining the introduction of non-separated manifolds in the study of quotients.

Example. Let us study the classification by translations of oriented closed segments (a, b) in a real line (with at least a finite end-point, and eventually $a = b$). These segments may be identified with points of $\mathcal{V} = \mathbb{P}_1 \times \mathbb{P}_1 - \{(\infty, \infty)\}$, which is a separated smooth manifold. If we consider the additive group $\mathbb{G}_a = (\mathbb{R}, +)$ of translations, we have a differentiable action of $\mathbb{G}_a$ on the manifold $\mathcal{V}$ of all oriented segments: $\lambda + (a, b) := (\lambda + a, \lambda + b)$. The orbits of this action are:

1. Segments (a, ∞), with a finite, form a orbit.
2. Segments (∞, a), with a finite, form a orbit.
3. Bounded segments (a, b), with $b - a = D$ a given constant, form a orbit.

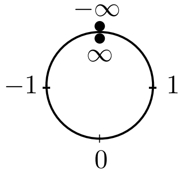
Therefore, if we extend D with the convention

$$D(a, \infty) = +\infty \quad , \quad D(\infty, a) = -\infty \ ,$$

we obtain the quotient of $\mathcal{V}$ by $\mathbb{G}_a$,

$$D : \mathcal{V} \longrightarrow \widetilde{\mathbb{P}}_1 \ ,$$

where $\widetilde{\mathbb{P}}_1$ stands for a real projective line where the infinity point is split (see 1.3) in two points $\pm\infty$:

$\widetilde{\mathbb{P}}_1 =$ 

From a categorial point of view, it may be reasonable to consider the notion of categorial quotient: Given a differentiable action of a Lie group G on a smooth manifold $\mathcal{V}$, a differentiable map $\pi\colon \mathcal{V} \to \mathcal{C}$ to a smooth manifold $\mathcal{C}$ is said to be the **categorial quotient** if for any differentiable map $\varphi\colon \mathcal{V} \to \mathcal{W}$ to a smooth manifold, which is constant on the orbits, there exists a unique differentiable map $\psi\colon \mathcal{C} \to \mathcal{W}$ such that $\varphi = \psi \circ \pi$. Of course, the quotient manifold $\pi\colon \mathcal{V} \to \mathcal{V}/G$, if it exists, coincides with the categorial quotient. However, the categorial quotient exists in some situations where the quotient manifold does not. In such a case, the categorial quotient looks inadequate from a geometric point of view.

Example. Let us consider the multiplicative action of $G = \mathbb{R}_+$ on the complex plane $\mathbb{C}$. The topological quotient space $\mathbb{C}/G$ has a unique closed point $[0]$, so that it is not a T_1-space; in particular, $\mathbb{C}/G$ is not a smooth manifold. In this case the categorical quotient of $\mathbb{C}$ by G exists, since it is the projection $\mathbb{C} \to p$ onto the one-point manifold p.

11.2 Equivariant Embedding Theorem

Definition. We say that a continuous action $\mu\colon G \times E \to E$ of a Lie group G on a topological vector space E is **linear**, or that G acts by linear automorphisms, if the maps $\mu_g\colon E \to E$, $\mu_g(e) = g \cdot e$, are linear.

If E is a finite-dimensional real vector space, then continuous linear actions of G on E are just continuous linear representations (continuous morphisms of groups) $\varrho\colon G \to Gl(E)$, where $\varrho(g)(e) := g \cdot e$.

Proposition 11.6. *Let G be a Lie group and let E be a finite-dimensional real vector space. Any continuous linear representation $\varrho\colon G \to Gl(E)$ is differentiable, hence the corresponding action $\mu\colon G \times E \to E$, $\mu(g, e) = \varrho(g)(e)$, is differentiable.*

Proof. In fact, the graph $\Gamma_\varrho = \{(g, \varrho(g))\colon g \in G\} \subset G \times Gl(E)$ is a closed subgroup and it is homeomorphic to G; hence it is a smooth submanifold of the same dimension as G ([73] 3.42). The first projection $\pi_1\colon \Gamma_\varrho \to G$, being a morphism of groups, has constant rank; hence it is a diffeomorphism (1.23). We conclude that the linear representation $\varrho = \pi_2 \pi_1^{-1}$ is differentiable.

□

Definitions. A **differentiable action** of a Lie group G on a differentiable space X is defined to be any morphism of differentiable spaces $\mu\colon G \times X \to X$ such that the following diagrams are commutative:

$$\begin{array}{ccc} G\times G\times X & \xrightarrow{m\times Id} & G\times X \\ \Big\downarrow{\scriptstyle Id\times\mu} & & \Big\downarrow{\scriptstyle \mu} \\ G\times X & \xrightarrow{\ \mu\ } & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{1\times Id} & G\times X \\ & {\scriptstyle Id}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & X & \end{array}$$

where $m\colon G{\times}G \to G$, $m(g_1,g_2)=g_1g_2$, is the operation of the Lie group G.

Of course, when X is a reduced differentiable space, the commutative character of the above diagrams is equivalent to the usual conditions

$$(g_1g_2)\cdot x = g_1\cdot(g_2\cdot x) \qquad \text{and} \qquad 1\cdot x = x\ .$$

Given differentiable actions $\mu_X\colon G{\times}X \to X$ and $\mu_Y\colon G{\times}Y \to Y$ of G on two differentiable spaces X, Y, we say that a morphism of differentiable spaces $\phi\colon Y \to X$ is G**-equivariant** if the following square is commutative:

$$\begin{array}{ccc} G\times Y & \xrightarrow{\mu_Y} & Y \\ {\scriptstyle Id\times\phi}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi} \\ G\times X & \xrightarrow{\mu_X} & X \end{array}$$

Let $\mu\colon G{\times}X \to X$ be a differentiable action and let $i\colon Y \hookrightarrow X$ be a differentiable subspace. If the morphism $G{\times}Y \xrightarrow{Id\times i} G{\times}X \xrightarrow{\mu} X$ factors through Y, then Y is said to be G**-invariant**. In such a case, the induced morphism $G{\times}Y \to Y$ is a differentiable action and the inclusion morphism $i\colon Y \hookrightarrow X$ is G-equivariant.

Definition. Let $\mu\colon G\times A \to A$ be a continuous action of a Lie group G on a topological algebra A. We say that G **acts on** A **by automorphisms of algebras** if the maps $\mu_g\colon A \to A$, $\mu_g(a) := g\cdot a$, are automorphisms of $\mathbb{R}$-algebras.

Example. Let $\mathcal{C}(G,\mathbb{R})$ be the algebra of all real-valued continuous functions on a Lie group G. We have a natural continuous action of G on $\mathcal{C}(G,\mathbb{R})$ by automorphisms of algebras: $(g\cdot f)(g') := f(g^{-1}g')$.

Example. Let $\mu\colon G{\times}E \to E$ be a continuous linear action of a Lie group G on a finite-dimensional real vector space E. Then we have a natural continuous action of G on $\mathcal{C}^\infty(E)$ by automorphisms of algebras: $(g\cdot f)(e) := f(g^{-1}\cdot e)$.

Let $\mu\colon G\times X \to X$ be a differentiable action of a Lie group G on an affine differentiable space X. If $g\in G$, then the composition

$$X \simeq g{\times}X \hookrightarrow G{\times}X \xrightarrow{\mu} X$$

is an isomorphism $g\colon X \simeq X$, and it defines an isomorphism of $\mathbb{R}$-algebras $g^*\colon \mathcal{O}_X(X) \simeq \mathcal{O}_X(X)$. So we obtain an action of G on the algebra of differentiable functions $\mathcal{O}_X(X)$ by automorphisms of algebras:

$$g{\cdot}f := (g^{-1})^*f \qquad , \qquad g\in G,\ f\in\mathcal{O}_X(X)\ .$$

Note that, by definition, $g\cdot f$ is the restriction of μ^*f to $g^{-1}{\times}X \simeq X$. The following lemma shows that this action is continuous.

Lemma 11.7. *Let $\mu\colon G\times X\to X$ be a differentiable action of a Lie group G on an affine differentiable space X. The corresponding action of G on $\mathcal{O}_X(X)$ is continuous.*

Proof. For any affine differentiable space Y, we write $\mathcal{O}(Y):=\mathcal{O}_Y(Y)$. Let us we consider the map

$$\delta_Y:\ G\times\mathcal{O}(G\times Y)\longrightarrow\mathcal{O}(Y)$$

such that $\delta_Y(g,f)$ is the restriction of f to $g\times Y\simeq Y$. It is easy to check that $\delta_{\mathbb{R}^n}$ is continuous. Let $\mathcal{O}(X)=\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$; we have a commutative square

$$\begin{array}{ccc} G\times\mathcal{O}(G\times\mathbb{R}^n) & \xrightarrow{\delta_{\mathbb{R}^n}} & \mathcal{O}(\mathbb{R}^n)=\mathcal{C}^\infty(\mathbb{R}^n) \\ \downarrow & & \downarrow \\ G\times\mathcal{O}(G\times X) & \xrightarrow{\delta_X} & \mathcal{O}(X) \end{array}$$

where the vertical maps are surjective continuous open maps. Since $\delta_{\mathbb{R}^n}$ is continuous, it follows that δ_X is continuous.

Given a differentiable action $\mu\colon G\times Y\to Y$, we conclude that the action of G on $\mathcal{O}_X(X)$ is continuous, because it is the composition of the following continuous maps:

$$G\times\mathcal{O}(X)\xrightarrow{(id)\times\mu^*}G\times\mathcal{O}(G\times X)\xrightarrow{(inv)\times(id)}G\times\mathcal{O}(G\times X)\xrightarrow{\delta_X}\mathcal{O}(X)\,.$$

□

Lemma 11.8. *Let $\mu\colon G\times X\to X$ and $\mu\colon G\times Y\to Y$ be differentiable actions of a Lie group G on two affine differentiable spaces X and Y. A morphism $\varphi\colon X\to Y$ is G-equivariant if and only if the morphism $\varphi^*\colon\mathcal{O}_Y(Y)\to\mathcal{O}_X(X)$ is G-equivariant.*

Proof. If $\varphi\colon X\to Y$ is G equivariant, then the following square

$$\begin{array}{ccc} g\times X & \xrightarrow{\mu} & X \\ {\scriptstyle Id\times\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ g\times Y & \xrightarrow{\mu} & Y \end{array}$$

is commutative for any $g\in G$. Equivalently, the square

$$\begin{array}{ccc} \mathcal{O}(X) & \xleftarrow{g^*} & \mathcal{O}(X) \\ {\scriptstyle\varphi^*}\uparrow & & \uparrow{\scriptstyle\varphi^*} \\ \mathcal{O}(Y) & \xleftarrow{g^*} & \mathcal{O}(Y) \end{array}$$

is commutative for any $g\in G$, that is to say, $\varphi^*\colon\mathcal{O}_Y(Y)\to\mathcal{O}_X(X)$ is G-equivariant.

Conversely, if $\varphi^*\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is G-equivariant, then reversing the above argument we obtain that the square

$$\begin{array}{ccc} g\times X & \xrightarrow{\mu} & X \\ {\scriptstyle Id\times\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ g\times Y & \xrightarrow{\mu} & Y \end{array}$$

is commutative for any $g \in G$. According to 7.16, so is commutative the square

$$\begin{array}{ccc} G\times X & \xrightarrow{\mu} & X \\ {\scriptstyle Id\times\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ G\times Y & \xrightarrow{\mu} & Y \end{array}$$

that is to say, φ is G-equivariant.

□

Definition. Let $\theta\colon G \times E \to E$ be a continuous linear action of a compact Lie group G on a topological vector space E. A vector $e \in E$ is said to be a **representation** vector if the vector subspace of E generated by the orbit Ge is finite-dimensional. Peter-Weyl theorem (see [46]) states that continuous representation functions are dense in $\mathcal{C}(G, \mathbb{R})$.

Theorem 11.9. *For any continuous linear action of a compact Lie group G on a Fréchet vector space F, representation vectors are dense in F.*

Proof. Given a vector $f \in F$, a continuous seminorm q on F, and a positive real number ε, we have to prove the existence of some representation vector $f_v \in F$ such that $q(f - f_v) < \varepsilon$.

For any continuous function $v \in \mathcal{C}(G, \mathbb{R})$ we shall put

$$f_v := \int_G v(g)(g \cdot f)\, \mathrm{d}g\ ,$$

where $\mathrm{d}g$ is the unique left-invariant measure on G such that $\int_G \mathrm{d}g = 1$ (see [54] for the integration of continuous functions on a compact space with values in a Fréchet space). Take $s = \max\{q(g \cdot f)\colon\ g \in G\}$.

Since the map $\mu_f\colon G \to F$, $\mu_f(g) = g \cdot f$ is continuous, it follows the existence of a neighbourhood U of the identity of G such that

$$q(f - gf) < \varepsilon/2$$

for any $g \in U$. Let u be a non-negative continuous function, with support K contained in U, such that $\int u\, \mathrm{d}g = 1$. We have

$$\begin{aligned} q(f - f_u) &= q\Big(\int_G u(g)(f - gf)\, \mathrm{d}g\Big) \ \leq\ \int_G u(g)q(f - gf)\, \mathrm{d}g \\ &= \int_K u(g)q(f - gf)\, \mathrm{d}g\ <\ \int_K \frac{\varepsilon u}{2}\, \mathrm{d}g\ =\ \frac{\varepsilon}{2}\,. \end{aligned}$$

Now, by Peter-Weyl theorem, there exists a continuous representation function $v \in \mathcal{C}(G, \mathbb{R})$ such that $|u(g) - v(g)| < \varepsilon/2s$ for any $g \in G$, so that

$$q(f - f_v) \leq q(f - f_u) + q(f_u - f_v) \ < \ \frac{\varepsilon}{2} + q\Big(\int_G (u(g) - v(g))(gf)\,\mathrm{d}g\Big)$$
$$\leq \frac{\varepsilon}{2} + \int_G |u(g) - v(g)|\, q(gf)\mathrm{d}g \ < \ \frac{\varepsilon}{2} + \int_G \frac{\varepsilon s}{2s}\,\mathrm{d}g \ = \ \varepsilon\ .$$

Moreover, since v is a representation continuous function on G, we have $Gv \subseteq \mathbb{R}v_1 + \ldots + \mathbb{R}v_n$ for some $v_1, \ldots, v_n \in \mathcal{C}(G, \mathbb{R})$. Therefore, if $h \in G$, then

$$h \cdot f_v = h \cdot \Big(\int_G v(g)(g \cdot f)\,\mathrm{d}g\Big) = \int_G v(g)(hg \cdot f)\,\mathrm{d}g$$
$$= \int_G v(h^{-1}hg)(hg \cdot f)\,\mathrm{d}g = \int_G v(h^{-1}g)(g \cdot f)\,\mathrm{d}g \quad (\mathrm{d}g \text{ is left-invariant})$$
$$= \int_G (h \cdot v)(g)(g \cdot f)\,\mathrm{d}g = f_{h \cdot v} \in \mathbb{R}f_{v_1} + \ldots + \mathbb{R}f_{v_n}\ .$$

We conclude that $f_v \in F$ is a representation vector and $q(f - f_v) < \varepsilon$.

□

Now we extend Mostow's theorem ([32] 7.4) on the equivariant embedding of smooth manifolds to the case of differentiable spaces.

Equivariant embedding theorem ([11]). *Let X be a compact separated differentiable space and let G be a compact Lie group. If $\mu\colon G \times X \to X$ is a differentiable action, then there exists a differentiable linear representation $\varrho\colon G \to Gl(E)$ and a G-equivariant closed embedding $j\colon X \hookrightarrow E$.*

Proof. According to 5.28, any compact separated differentiable space is affine: $X = \mathrm{Spec}_r\, A$. Let us put $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ and then $X = \mathrm{Spec}_r\, A = (\mathfrak{a})_0$ is a compact subset of $\mathbb{R}^n$. By 11.7, the action of G on A induced by μ is continuous. Representation functions are dense in A, because A is a Fréchet space and we may apply theorem 11.9; hence there are representation functions $f_1, \ldots, f_n \in A$ so close to the cartesian coordinates $x_1, \ldots, x_n$ that $\mathrm{d}_x f_1, \ldots, \mathrm{d}_x f_n$ span the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ at any point $x \in X$. Hence $f_1, \ldots, f_n$ separate infinitely near points in X. By 5.21, these representation functions separate points in a neighbourhood of each point of X (but it may be that $f_1, \ldots, f_n$ do not separate points of X). Let us consider a finite open cover of X by such neighbourhoods and let ε be a positive number such that, whenever the distance of two points of X is smaller than ε, both points are in some member of the cover. Let $h_1, \ldots, h_n \in A$ be representation functions so close to $x_1, \ldots, x_n$ that, if $h_i(x) = h_i(y)$ for any $i = 1, \ldots, n$, then $d(x, y) < \varepsilon$, so that $f_j(x) \neq f_j(y)$ for some index j. Therefore $f_1, \ldots, f_n, h_1, \ldots, h_n$ separate points of X.

Let F be the finite-dimensional vector subspace of A generated by the orbits

$$Gf_1\ ,\ \ldots\ ,\ Gf_n\ ,\ Gh_1\ ,\ \ldots\ ,\ Gh_n\ ,$$

which is G-invariant, and let E be the dual space of F. According to 2.20, the inclusion map $F \to A$ may be extended, in a unique way, so as to obtain a morphism of algebras $\varphi^*\colon \mathcal{C}^\infty(E) \to A$. This morphism is G-equivariant by the uniqueness of the extension, and it is surjective because $f_1, \ldots, f_n$, $h_1, \ldots, h_n$ separate points of $X = \operatorname{Spec}_r A$ and infinitely near points (5.25). Finally, the epimorphism φ^* defines a closed embedding $\varphi\colon X \hookrightarrow E$, which is G-equivariant by lemma 11.8.

□

Definition. Given a differentiable action of a Lie group G on a differentiable space X, the **type of isotropy** τ_x of a point $x \in X$ is defined to be the conjugacy class of the **isotropy subgroup** $I_x := \{g \in G\colon\ g \cdot x = x\}$, and we put $\tau_y \leq \tau_x$ whenever $gI_yg^{-1} \subseteq I_x$ for some $g \in G$. It is an order relation when the isotropy subgroups are compact ([4] IX.9 Lemma 6).

Corollary 11.10. *Given a differentiable action of a compact Lie group G on a compact separated differentiable space X, the number of types of isotropy is finite, and each point $x \in X$ has a neighbourhood U such that $\tau_y \leq \tau_x$ for any $y \in U$.*

Proof. Any finite-dimensional differentiable linear representation of a compact Lie group has both properties ([32] 7.2 and [4] IX.9 Th. 2.a). Now the result follows directly from the Equivariant embedding theorem.

□

11.3 Geometric Quotient

Definition. Any differentiable action $\mu\colon G \times X \to X$ of a Lie group G on a differentiable space X defines an equivalence relation on the underlying topological space X, and we have a sheaf of $\mathbb{R}$-algebras $\mathcal{O}_{X/G}$ on the quotient topological space X/G:

$$\mathcal{O}_{X/G}(V) \ := \ \mathcal{O}_X(\pi^{-1}V)^G$$

where $\pi\colon X \to X/G$ denotes the canonical projection. The **geometric quotient** of X by G is defined to be the morphism of locally ringed spaces

$$(\pi, \pi^*)\colon (X, \mathcal{O}_X) \ \longrightarrow \ (X/G, \mathcal{O}_{X/G})$$

where π^* is the morphism of sheaves defined by the inclusion morphisms $\mathcal{O}_{X/G}(V) = \mathcal{O}_X(\pi^{-1}V)^G \subseteq \mathcal{O}_X(\pi^{-1}V)$.

The geometric quotient is a locally ringed $\mathbb{R}$-space and the fundamental question is to determine whether it is a differentiable space.

Example. Let us consider a differentiable action of a Lie group G on a smooth manifold $\mathcal{V}$. If $\varphi\colon \mathcal{V} \to \overline{\mathcal{V}}$ is a surjective submersion whose fibres coincide (set-theoretically) with the orbits of the action, then $\mathcal{V}/G$ is a smooth manifold and φ induces a diffeomorphism $\mathcal{V}/G = \overline{\mathcal{V}}$ (see 11.1).

The geometric quotient is a local concept in the following sense: If U is a G-invariant open subspace of X, then $\pi(U)$ is open in X/G and the morphism of ringed spaces $(\pi,\pi^*)|_U\colon U \to (\pi(U), \mathcal{O}_{X/G}|_{\pi(U)})$ is the geometric quotient of the action of G on U. Therefore, if X admits a cover $\{U_i\}$ by G-invariant open subspaces such that U_i/G is a differentiable space for any index i, then X/G is a differentiable space.

It is immediate to check that the geometric quotient $\pi\colon X \to X/G$ is the cokernel of the morphisms $\mu, p_2\colon (G\times X) \rightrightarrows X$ in the category of locally ringed $\mathbb{R}$-spaces (even in the category of arbitrary ringed spaces). That is to say,

Proposition 11.11. *Let $\mu\colon G\times X \to X$ be a differentiable action of a Lie group G on a differentiable space X. Let $\varphi\colon X \to Y$ be a morphism of ringed spaces such that $\varphi\circ\mu = \varphi\circ p_2$. Then there exists a unique morphism of ringed spaces $\overline{\varphi}\colon X/G \to Y$ such that $\varphi = \overline{\varphi}\circ\pi$.*

Proposition 11.12. *Let $\mu\colon G\times X \to X$ be a differentiable action of a Lie group G on a differentiable space X. Given a point $x\in X$, let us assume that the orbit Gx is a locally closed subset of X. Then we have the following facts:*

1. *The orbit Gx, considered as a differentiable subspace of X endowed with the reduced structure, is a smooth manifold.*
2. *The natural bijection $G/I_x \to Gx$, $[g]\mapsto g\cdot x$, is a diffeomorphism.*

Proof. The isotropy subgroup I_x is closed in G (hence it is a Lie subgroup) because it is the fibre of the morphism $\varphi\colon G \xrightarrow{\cdot x} X$ over x. By 11.3, we know that G/I_x is a smooth manifold.

By the categorial property of the geometric quotient (11.11), the morphism $\varphi\colon G \xrightarrow{\cdot x} Gx \subseteq X$ induces a bijective morphism $\varphi\colon G/I_x \xrightarrow{\cdot x} Gx$. To prove that this bijection is an isomorphism, it is enough to show that $\varphi\colon G/I_x \xrightarrow{\cdot x} Gx$ is a local embedding, since both differentiable spaces are reduced. The rank of the linear tangent map φ_* is constant along G/I_x because φ is G-equivariant. Now, the problem being local, we may assume that Gx is affine: $Gx\subseteq\mathbb{R}^n$. Finally, it is a standard fact that any injective morphism $\varphi\colon G/I_x \to Gx \hookrightarrow \mathbb{R}^n$ between smooth manifolds, whose linear tangent map φ_* has constant rank, is a local embedding (see 1.23).

□

Let us consider a differentiable action of a compact Lie group G on a differentiable space X. If f is a differentiable function on a G-invariant affine open subspace U, then the map $G\to\mathcal{O}_X(U)$, $f\mapsto g\cdot f$, is continuous by lemma 11.7, and averaging over G

$$\tilde{f} := \int_G (g\cdot f)\,\mathrm{d}g$$

we obtain a G-invariant differentiable function $\tilde{f}$ on U, and $\tilde{f} = f$ whenever f is G-invariant. Recall that $\mathrm{d}g$ denotes the unique left-invariant measure on G such that $\int_G \mathrm{d}g = 1$.

Hilbert's Finiteness Theorem. *Let $G \to Gl(\mathbb{R}^n)$ be a differentiable linear representation of a compact Lie group G. The $\mathbb{R}$-algebra of G-invariant polynomials on $\mathbb{R}^n$ is finitely generated, i.e., there exist G-invariant homogeneous polynomials $p_1, \dots, p_k$ on $\mathbb{R}^n$ such that*

$$\mathbb{R}[x_1, \dots, x_n]^G = \mathbb{R}[p_1, \dots, p_k] .$$

Proof. Let I be the ideal of $\mathbb{R}[x_1, \dots, x_n]$ generated by all G-invariants homogeneous polynomials of positive degree. Since any ideal of $\mathbb{R}[x_1, \dots, x_n]$ is finitely generated (by Hilbert's basis theorem), we can find finitely many homogeneous G-invariant generators $p_1, \dots, p_k$ of I. Now, we show by induction on d that every homogeneous invariant polynomial p of degree d lies in $\mathbb{R}[p_1, \dots, p_k]$.

The case $d = 0$ is trivial. Suppose $d > 0$. Then $p \in I$ and we can write it in the form

$$p = q_1 p_1 + \cdots + q_k p_k .$$

Averaging over G, we obtain that

$$p = \tilde{q}_1 p_1 + \cdots + \tilde{q}_k p_k$$

where $\tilde{q}_1, \dots, \tilde{q}_k$ are invariant polynomials. Since we can replace each $\tilde{q}_i$ by its homogeneous part of degree $d - \deg(p_i)$, we may assume that $\tilde{q}_i$ is homogeneous of degree $< d$. Hence, by induction $\tilde{q}_1, \dots, \tilde{q}_k \in \mathbb{R}[p_1, \dots, p_k]$ and we conclude that $p \in \mathbb{R}[p_1, \dots, p_k]$.

□

A similar result is valid for differentiable functions in the following sense:

Schwarz's Theorem. *Let $G \to Gl(\mathbb{R}^n)$ be a differentiable linear representation of a compact Lie group G, let $p_1, \dots, p_k$ generators of $\mathbb{R}[x_1, \dots, x_n]^G$ and let $p = (p_1, \dots, p_k)\colon \mathbb{R}^n \to \mathbb{R}^k$. Then*

$$p^* \mathcal{C}^\infty(\mathbb{R}^k) = \mathcal{C}^\infty(\mathbb{R}^n)^G .$$

See [45, 56] for a proof of Schwarz's theorem. This result has been extended to the case of representations of reductive groups (see [25]).

Now, we shall reformulate Schwarz's theorem in the context of differentiable spaces. Firstly, we need the following lemma.

Lemma 11.13. *With the notations of the Schwarz's theorem, the map*

$$p\colon \mathbb{R}^n \longrightarrow \mathbb{R}^k \quad , \quad p(x) = (p_1(x), \dots, p_k(x)) ,$$

is proper and it induces a homeomorphism $\mathbb{R}^n/G \simeq Z$ onto a closed subset $Z := p(\mathbb{R}^n)$ in $\mathbb{R}^k$.

Proof. Let us consider a G-invariant metric on $\mathbb{R}^n$ (obtained by averaging any metric). Let r^2 be the square of the radius function on $\mathbb{R}^n$ with respect to the given metric. It is clear that r^2 is a G-invariant polynomial, hence it is a polynomial in the p_i's. Since r^2 is a proper map, so is p.

Now, let us show that the continuous map $p : \mathbb{R}^n/G \to \mathbb{R}^k$ is injective or, equivalently, that $\mathbb{R}[p_1, \dots, p_k]$ separates the orbits of G. Since G is compact, the orbits are closed subsets of $\mathbb{R}^n$. Given two different orbits Y and Y', let f be a differentiable function on $\mathbb{R}^n$ such that $f(Y) = 0$ and $f(Y') = 1$. Given a small number $\varepsilon > 0$, since polynomials are dense in $\mathcal{C}^\infty(\mathbb{R}^n)$, we can find a polynomial q so close to f that $|q(y)| \leq \varepsilon$ and $|q(y') - 1| < \varepsilon$ for any $y \in Y, y' \in Y'$. Averaging over G, we obtain a G-invariant polynomial $\tilde{q}$ taking different values at Y and at Y'.

Finally, the continuous bijection $p\colon \mathbb{R}^n/G = Z$ is a homeomorphism because both spaces are locally compact and Hausdorff.

□

Let us consider on Z the structure of reduced differentiable space induced by $\mathbb{R}^k$, that is to say, any differentiable function on Z is the restriction of some differentiable function on $\mathbb{R}^k$. Schwarz's theorem may be re-stated in the following terms:

Theorem 11.14. *Let $G \to Gl(\mathbb{R}^n)$ be a differentiable linear representation of a compact Lie group G. With the previous notations, we have:*

1. *The geometric quotient $\mathbb{R}^n/G$ is a reduced affine differentiable space.*
2. *The bijection $p\colon \mathbb{R}^n/G = Z$ is an isomorphism of differentiable spaces.*

Proof. Given the identification $p\colon \mathbb{R}^n/G = Z$, let us prove that $\mathcal{O}_{\mathbb{R}^n/G} = \mathcal{C}^\infty_Z$, so that the geometric quotient $(\mathbb{R}^n/G, \mathcal{O}_{\mathbb{R}^n/G})$ coincides with the reduced differentiable space $(Z, \mathcal{C}^\infty_Z)$, which is affine by 5.8.

The inclusion $\mathcal{C}^\infty_Z(V) \subseteq \mathcal{O}_{\mathbb{R}^n/G}(V)$ is clear for any open subset $V \subseteq Z$.

Conversely, given $f \in \mathcal{O}_{\mathbb{R}^n/G}(V) = \mathcal{C}^\infty(p^{-1}V)^G$, we have to show the existence of a differentiable function $F \in \mathcal{C}^\infty_Z(V)$ such that $f = F(p_1, \dots, p_k)$. By the Localization theorem for differentiable functions, any differentiable function f on $p^{-1}V$ is a quotient $f = u/v$, where $u, v \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $v > 0$ on $p^{-1}V$. Averaging over G we obtain

$$f = \tilde{u}/\tilde{v} \quad , \qquad \tilde{u}, \tilde{v} \in \mathcal{C}^\infty(\mathbb{R}^n)^G \quad , \quad \tilde{v} > 0 \text{ on } p^{-1}V ,$$

since $\widetilde{fh} = f\tilde{h}$ whenever f is G-invariant. According to Schwarz's theorem, we have $\tilde{u} = P(p_1, \dots, p_k)$, $\tilde{v} = Q(p_1, \dots, p_k)$ for some differentiable functions $P, Q \in \mathcal{C}^\infty(\mathbb{R}^k)$, and it is clear that $Q > 0$ on V. We conclude that P/Q defines a differentiable function $F \in \mathcal{C}^\infty(V)$ such that

$$F(p_1, \dots, p_k) \;=\; \frac{P(p_1, \dots, p_k)}{Q(p_1, \dots, p_k)} \;=\; \frac{\tilde{u}}{\tilde{v}} \;=\; f \,.$$

□

Lemma 11.15. *Let $\mu\colon G \times X \to X$ be a differentiable action of a compact Lie group on an affine differentiable space and let Y be a closed G-invariant differentiable subspace of X. If the geometric quotient X/G is a differentiable space, then so is Y/G and the natural morphism $Y/G \hookrightarrow X/G$ is a closed embedding.*

Proof. It is clear that $\pi(Y)$ is closed in X/G and that it is homeomorphic to Y/G. By 5.9, we only have to show that, for any affine open subspace $U \subseteq X/G$, the restriction morphism

$$\mathcal{O}_{X/G}(U) = \mathcal{O}_X(\pi^{-1}U)^G \longrightarrow \mathcal{O}_Y(Y \cap \pi^{-1}U)^G = \mathcal{O}_{Y/G}(U \cap \pi(Y))$$

is surjective, because its kernel is a closed ideal, since so is the kernel of the restriction morphism $\mathcal{O}_X(\pi^{-1}U) \to \mathcal{O}_Y(Y \cap \pi^{-1}U)$. Now, if $f \in \mathcal{O}_Y(Y \cap \pi^{-1}U)^G$, then $f = h|_{Y\cap\pi^{-1}U}$ for some $h \in \mathcal{O}_X(\pi^{-1}U)$, because $Y \cap \pi^{-1}U$ is a closed differentiable subspace of the affine open subspace $\pi^{-1}(U)$. Hence $\tilde{h} \in \mathcal{O}_X(\pi^{-1}U)^G$ and $f = h|_{Y\cap\pi^{-1}U} = \tilde{h}|_{Y\cap\pi^{-1}U}$ since h is G-invariant on $Y \cap \pi^{-1}U$.

□

Remark 11.16. Given a point $[x] \in X/G$, let us consider the fibre $F = \pi^{-1}[x]$ endowed with the natural (eventually non-reduced) differentiable structure. Then F/G is the one-point differentiable space. In other words, if A is the ring of the fibre, then $A^G = \mathbb{R}$.

Proof. Since the composition morphism $F/G \to [x] \hookrightarrow X/G$ is an embedding, the composition map $\mathcal{O}(X/G) \to \mathbb{R} \to A^G$ is surjective; hence $\mathbb{R} = A^G$.

□

Theorem 11.17. *Given a differentiable action of a compact Lie group G on a separated differentiable space X, the geometric quotient X/G is a separated differentiable space.*

Proof. First we prove that X/G is a differentiable space. Since it is a local problem and any point $x \in X$ has a G-invariant compact neighbourhood (GK, where K is any compact neighbourhood), we may assume that X is compact. In such a case, by the Equivariant embedding theorem, there exists a finite-dimensional differentiable linear representation lineal $G \to Gl(E)$ and a G-equivariant closed embedding $X \hookrightarrow E$. From 11.14 and 11.15 we conclude that X/G is a differentiable space

Let us show that X/G is separated. Given points $[x] \neq [y]$ in X/G, let K_x be a compact neighbourhood of x in X such that $K_x \cap Gy = \emptyset$ (note that the orbit Gy is a closed subset of X since it is compact). Taking GK_x instead of K_x, we may assume that K_x is G-invariant. Then the complement of K_x is a G-invariant open neighbourhood of y in X; hence $[x]$ and $[y]$ admit disjoint neighbourhoods and we conclude that X/G is separated.

□

Remark 11.18. Let us consider a differentiable action of a finite group G on an affine differentiable space $X \subseteq \mathbb{R}^n$. Then the quotient map $\pi\colon X \to X/G$ is a finite morphism.

Proof. By Theorem 11.17 we know that X/G is a separated differentiable space. It is immediate that π is a closed separated map, so we only have to prove that the fibre $F = \pi^{-1}[x]$ is a finite differentiable space of bounded degree for any $[x] \in X/G$. Let A be the differentiable algebra of all differentiable functions on F. Since F is a closed differentiable subspace of $X \subseteq \mathbb{R}^n$, we may write $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ for certain closed ideal $\mathfrak{a}$. Let us show that A is a finite algebra of degree $\leq r^n$, where r is the order of G. By 11.16, we have that $A^G = \mathbb{R}$. This fact implies that the morphism $\mathbb{R} \to A$ is integral, since any element $a \in A$ satisfies a relation of integral dependence

$$\prod_{g \in G} (t - g \cdot a) = t^r + \lambda_1 t^{r-1} + \cdots + \lambda_r = 0$$

whose coefficients λ_i are clearly G-invariants, i.e., $\lambda_i \in \mathbb{R}$ by 11.16. In particular, each coordinate function x_i satisfies some equation $P_i(x_i) = 0$ in A, where $P_i(t)$ is a polynomial of degree r. Therefore, A is a quotient of the $\mathbb{R}$-algebra $\mathcal{C}^\infty(\mathbb{R}^n)/(P_1(x_1), \ldots, P_n(x_n))$. Since

$$\mathbb{R}[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n)) = \mathbb{R}[x_1]/P_1(x_1) \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathbb{R}[x_n]/P_n(x_n)$$

is a finite algebra of degree r^n, we conclude that

$$\mathcal{C}^\infty(\mathbb{R}^n)/(P_1(x_1), \ldots, P_n(x_n)) \overset{9.18}{=} [\mathbb{R}[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n))]_{\text{rat}}$$

is a finite algebra of degree $\leq r^n$, hence so is the quotient algebra A.

□

11.4 Examples

Let $G \times X \to X$ be a differentiable action of a Lie group G on a differentiable space X. If X/G is a differentiable space, then the orbits are closed subsets of X because they coincide topologically with the fibres of $\pi \colon X \to X/G$. In such a case, the orbits are smooth manifolds according to 11.12. However, the fibres of $\pi \colon X \to X/G$, with the natural differentiable structure, may be non-reduced. That is to say, fibres and orbits are not necessarily isomorphic as differentiable spaces. The following two simple examples show this phenomenon.

Example 11.19. Let $G = \{\pm 1\}$ acting multiplicatively on $\mathbb{R}$. Since the algebra of invariant polynomials is $\mathbb{R}[\,t\,]^G = \mathbb{R}[t^2]$, we know (11.14) that the geometric quotient is

$$\mathbb{R} \longrightarrow [0, \infty) \qquad , \qquad t \mapsto t^2 \ .$$

The fibre of any point $a \in [0, \infty)$ is the subspace $t^2 = a$ in $\mathbb{R}$. In particular, the fibre of the origin 0 is the subspace $t^2 = 0$, defined by the differentiable algebra $\mathcal{C}^\infty(\mathbb{R})/(t^2)$, whereas the corresponding orbit is $t = 0$, i.e., the fibre counts twice the orbit.

Example 11.20. Let G be the group of rotations of $\mathbb{R}^2$ with centre at the origin. The algebra of invariant polynomials is $\mathbb{R}[x,y]^G = \mathbb{R}[x^2+y^2]$, hence the geometric quotient is

$$\mathbb{R}^2 \longrightarrow [0,\infty) \qquad , \qquad (x,y) \mapsto x^2+y^2 \ .$$

The ring of the fibre of the origin 0 is

$$\mathcal{C}^\infty(\mathbb{R}^2)/\,\overline{(x^2+y^2)} \ = \ \mathbb{R}[[x,y]]/(x^2+y^2) \ ,$$

whereas the corresponding orbit is the one-point space $\{0\} = \operatorname{Spec}_r \mathbb{R}$.

Example 11.21 (Quotients of smooth manifolds by finite groups). Let X be a smooth manifold and let G be a finite group of diffeomorphisms of X. The local structure of X/G may be understood as follows. Let us consider a G-invariant Riemannian metric on X (such a metric may be obtained from any metric on X by averaging with G). Given a point $x \in X$, let us consider the exponential map, which defines a diffeomorphism from the ϵ-ball of the tangent space T_xX to a small neighbourhood of x. Since the exponential map commutes with the action of the isotropy group I_x, it defines an isomorphism between a neighbourhood of $[x]$ in X/G and a neighbourhood of the origin in T_xX/I_x. Therefore, quotients $\mathbb{R}^n/G$, where G is a finite subgroup of the orthogonal group O_n, are local models for any quotient of a smooth manifold by a finite group.

Let us examine the case of dimension two. The only finite subgroups of O_2 (up to conjugation) are the following ones:

(i) The trivial subgroup $G = \{1\}$.

(ii) The subgroup $G = \mathbb{Z}/2\mathbb{Z}$ generated by the reflection in the y-axis.

(iii) The subgroup $G = \mathbb{Z}/n\mathbb{Z}$ generated by a rotation of order n.

(iv) The dihedral subgroup $G = D_n$ of order $2n$, with presentation

$$< a, b : a^2 = b^2 = (ab)^n = 1 > \ ,$$

whose generators a, b correspond to reflections in lines meeting at angle π/n.

Let us compute the geometric quotient $\mathbb{R}^2/G$ in the above four cases. Of course, in case (i) we have $\mathbb{R}^2/G = \mathbb{R}^2$.

Case (ii). When G is generated by the reflection in the y-axis, the algebra of invariant polynomials is $\mathbb{R}[x,y]^G = \mathbb{R}[x^2, y]$. By 11.14, the geometric quotient is the image of the morphism

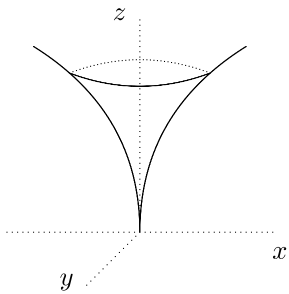
$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad , \qquad (x,y) \mapsto (x^2, y) \ .$$

Therefore, the geometric quotient $\mathbb{R}^2/G$ is the half-plane $x \geq 0$.

Case (iii). When G is generated by a rotation of order n, a lengthy computation gives the following generators for the algebra of invariant polynomials:

$$p_1(x,y) := \mathrm{Re}\,(x+iy)^n = \tfrac{1}{2}((x+iy)^n + (x-iy)^n)$$
$$p_2(x,y) := \mathrm{Im}\,(x+iy)^n = \tfrac{1}{2i}((x+iy)^n - (x-iy)^n),$$
$$p_3(x,y) := ||x+iy||^2 = x^2+y^2$$

By Theorem 11.14, the geometric quotient $\mathbb{R}^2/G$ is isomorphic to the image of the morphism $(p_1,p_2,p_3)\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, that is to say, $\mathbb{R}^2/G$ is isomorphic to the singular surface of $\mathbb{R}^3$ defined by the equations $x^2+y^2=z^n$, $z \geq 0$.



$z^n = x^2 + y^2$
$z \geq 0$

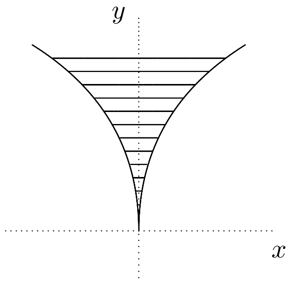
In this case, it is interesting to observe that the differentiable quotient does not coincide with the holomorphic quotient: Let us consider $\mathbb{R}^2 = \mathbb{C}$ as a complex manifold, then G acts on $\mathbb{C}$ by holomorphic automorphisms and the holomorphic quotient is the morphism $\varphi\colon \mathbb{C} \to \mathbb{C}$, $\varphi(z) = z^n$, since any G-invariant holomorphic function on $\mathbb{C}$ is an holomorphic function on the variable z^n.

By the categorial property of the geometric quotient, φ factors through a morphism $\mathbb{R}^2/G \to \mathbb{C}$. This morphism corresponds with the projection of the surface $x^2+y^2=z^n$, $z\geq 0$, onto the xy-plane.

Case (iv). When $G = D_n$, again a lengthy computation gives the following generators for the algebra of invariant polynomials:

$$p(x,y) := \mathrm{Re}\,(x+iy)^n = \tfrac{1}{2}((x+iy)^n + (x-iy)^n)$$
$$q(x,y) := ||x+iy||^2 = x^2+y^2$$

Therefore, the geometric quotient $\mathbb{R}^2/G$ is isomorphic to the image of the morphism $(p,q)\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, this is to say, $\mathbb{R}^2/G$ is isomorphic to the subset of $\mathbb{R}^2$ defined by the inequations $y^n \geq x^2$, $y \geq 0$.



$y^n \geq x^2$
$y \geq 0$

From the above description of the quotients $\mathbb{R}^2/G$, it is not difficult to prove that two quotients $\mathbb{R}^2/G_1$ and $\mathbb{R}^2/G_2$ (by finite subgroups of $Gl(\mathbb{R}^2)$) are isomorphic as differentiable spaces if and only if G_1 and G_2 are isomorphic groups. This type of differentiable space is closely related with the so called orbifolds (see [70] chap. 13). A 2-dimensional (differentiable) **orbifold** may be characterized as a differentiable space such that each point has a neighbourhood isomorphic to $\mathbb{R}^2/G$ for some finite subgroup $G \subseteq O_2$. We ignore whether this characterization is valid in higher dimensions (any differentiable orbifold is a differentiable space, but it is not clear whether the differentiable structure determines the full structure of the orbifold).

Example 11.22 (Symmetric powers of $\mathbb{R}$*).* Let us consider the action of the symmetric group S_n on $\mathbb{R}^n$ permuting the factors. It is well known that the $\mathbb{R}$-algebra of invariant polynomials is $\mathbb{R}[x_1,\dots,x_n]^{S_n} = \mathbb{R}[s_1,\dots,s_n]$, where $s_1,\dots,s_n$ are the elementary symmetric functions:

$$\begin{aligned} s_1 &:= x_1 + \cdots + x_n \\ s_r &:= \sum_{i_1<\cdots<i_r} x_{i_1}\cdots x_{i_r} \\ s_n &:= x_1\cdots x_n \end{aligned}$$

Therefore, the geometric quotient $\mathbb{R}^n/S_n$ is isomorphic to the image of the map

$$(s_1,\dots,s_n)\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n \ .$$

Note that $x_1,\dots,x_n$ are the roots of the polynomial $t^n - s_1t^{n-1} + \cdots \pm s_n$, hence a point $(a_1,\dots,a_n) \in \mathbb{R}^n$ belongs to the image of $(s_1,\dots,s_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ if and only if all the roots of $t^n - a_1t^{n-1} + \cdots + (-1)^n a_n$ are real.

For example, since $t^2 - a_1t + a_2$ has real roots if and only if the discriminant is non negative, $\Delta^2 := a_1^2 - 4a_2 \geq 0$, we obtain that $\mathbb{R}^2/S_2$ is isomorphic to the surface with boundary $x^2 - 4y \geq 0$ in $\mathbb{R}^2$.

Example 11.23 (Classification of r-jets of Riemannian metrics). Let $\mathcal{R}_n^r$ be the set of all r-jets of Riemannian metrics on $\mathbb{R}^n$ at the origin. Let G_{r+1} be the Lie group of all $(r+1)$-jets of local diffeomorphisms of $\mathbb{R}^n$ at the origin. We have a natural action of G_{r+1} on $\mathcal{R}_n^r$ and the quotient set $\mathcal{M}_n^r := \mathcal{R}_n^r/G_{r+1}$ is said to be the **moduli space** of r-jets of Riemannian metrics in dimension n.

The set $\mathcal{R}_n^r$ has a natural structure of smooth manifold and it may be proved that the geometric quotient $\mathcal{M}_n^r = \mathcal{R}_n^r/G_{r+1}$ is a differentiable space. Let us describe (without proofs) two elementary cases.

Case $n = 2$ and $r = 2$. Given the 2-jet j^2g (at the origin) of a Riemannian metric g on the plane $\mathbb{R}^2$, we may consider its curvature K_g at the origin. This assignment $j^2g \mapsto K_g$ defines an isomorphism $\mathcal{M}_2^2 = \mathbb{R}$.

Case $n = 3$ and $r = 2$. Given the 2-jet j^2g (at the origin) of a Riemannian metric g on $\mathbb{R}^3$, let us consider the Ricci endomorphism $Ricci_g\colon T_0\mathbb{R}^3 \to T_0\mathbb{R}^3$. Let us denote by

$$t^3 - s_1(g)t^2 + s_2(g)t - s_3(g)$$

the characteristic polynomial of $Ricci_g$. Its coefficients induce a closed embedding

$$(s_1, s_2, s_3)\colon \mathcal{M}_3^2 \hookrightarrow \mathbb{R}^3 \ .$$

Since the Ricci endomorphism is diagonalizable, any root of the characteristic polynomial is real. Now, the roots of a polynomial $t^3 - a_1 t^2 + a_2 t - a_3$ are real if and only if the discriminant

$$\Delta^2 := [(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)]^2 = -4a_1^3 a_3 - 27a_3^2 + 18a_1 a_2 a_3 - 4a_2^3 + a_1^2 a_2^2$$

is non-negative (where $\alpha_1, \alpha_2, \alpha_3$ denote the complex roots). Therefore, $\mathcal{M}_3^2$ is isomorphic to the differentiable subspace of $\mathbb{R}^3$ defined by the inequation

$$-4x^3 z - 27z^2 + 18xyz - 4y^3 + x^2 y^2 \geq 0 \ .$$

Taking a coordinate change $x = 3\bar{x}$, $y = \bar{y} + 3\bar{x}^2$, $z = \bar{z} + \bar{x}\bar{y} + \bar{x}^3$, we obtain the following simpler inequation for $\mathcal{M}_3^2$,

$$-27\bar{z}^2 - 4\bar{y}^3 \geq 0 \ .$$

The line $\bar{y} = \bar{z} = 0$ corresponds with the subset of (equivalence classes of) 2-jets of Riemannian metrics of constant curvature.

11.5 Quotients of Smooth Manifolds and Stratification

Definition. A continuous action $\mu\colon G \times X \to X$ is said to be **proper** when so is (see [3] I.10) the continuous map $\mu \times p_2\colon G \times X \to X \times X$. For example, if G is compact then any continuous action on a separated space is proper ([3] III, p.28, Prop.2).

In this Section we consider a differentiable proper action $\mu\colon G \times \mathcal{V} \to \mathcal{V}$ of a Lie group G on a paracompact (hence separated) smooth manifold $\mathcal{V}$. We shall prove that the geometric quotient $\mathcal{V}/G$ is a differentiable space and that it admits a locally finite stratification by locally closed submanifolds.

Let us recall some well–known facts about proper actions on paracompact smooth manifolds (see [4] IX.9).

Given $x \in \mathcal{V}$, the isotropy group I_x is a compact subgroup of G, the orbit Gx is a closed submanifold of $\mathcal{V}$ and there exists a diffeomorphism $G/I_x = Gx$.

Let $N \to Gx$ be the normal vector bundle of the orbit Gx in $\mathcal{V}$. The group G acts linearly on N and the projection $N \to Gx$ is G–equivariant. Moreover, there exits a G–invariant neighbourhood U of Gx in $\mathcal{V}$ and a G–equivariant diffeomorphism $\phi : U \to N$, which coincides with the zero section on Gx (see [4] IX.9 Prop. 6).

Theorem 11.24. *Let $G \times \mathcal{V} \to \mathcal{V}$ be a differentiable proper action of a Lie group G on a paracompact smooth manifold $\mathcal{V}$. Then the geometric quotient $\mathcal{V}/G$ is a reduced differentiable space.*

Proof. We use the previous notations. Since the problem is local, it is enough to show that N/G is a reduced differentiable space. Let N_x be the fibre of the normal bundle $N \to Gx$ over x. The isotropy group I_x is compact and it acts linearly on N_x; hence, by 11.14, the geometric quotient N_x/I_x is a reduced differentiable space. Then N/G also is a differentiable space according to the following lemma. □

Lemma 11.25. *The natural morphism $(N_x/I_x, \mathcal{O}_{N_x/I_x}) \longrightarrow (N/G, \mathcal{O}_{N/G})$ is an isomorphism of ringed spaces.*

Proof. The map $G \to Gx = G/I_x$, $g \to g \cdot x$, is a submersion, hence there exists a differentiable local section $g : W \to G$, where W is an open neighbourhood of x in Gx. Therefore $w = g(w) \cdot x$ for any $w \in W$ and we have a diffeomorphism

$$N_x \times W \ = \ N_{|W} \qquad (y, w) \mapsto g(w) \cdot y.$$

Now, it is clear that the natural map $N_x/I_x \to N/G$ is continuous and bijective. We conclude that this map is a homeomorphism, since for any I_x–invariant open subset $V \subseteq N_x$ we have $V = N_x \cap GV$, and GV is a G–invariant open subset of N (since $V \times W \subseteq GV$, we have that GV is a neighbourhood of V and it follows that GV is an open subset of N).

Finally, we have to prove that the restriction maps

$$\mathcal{O}_N(GV)^G \ \longrightarrow \ \mathcal{O}_{N_x}(V)^{I_x}$$

are bijective. They are clearly injective. They are also surjective: Given any I_x–invariant differentiable function $f : V \to \mathbb{R}$, let $f' : GV \to \mathbb{R}$ be the unique G–invariant extension of f. We only have to see that f' is differentiable. This function is differentiable on $V \times W$ since $f'(y, w) = f'(g(w) \cdot y) = f(y)$. Since f' is G–invariant, we conclude that it is differentiable on the entire GV. □

Now, we shall examine when the differentiable space $\mathcal{V}/G$ is a smooth manifold.

Lemma 11.26. *Let $G \to Gl(\mathbb{R}^n)$ be a differentiable linear representation of a compact Lie group G. Then $\mathbb{R}^n/G$ is a smooth manifold if and only if G acts on $\mathbb{R}^n$ by the identity.*

Proof. If G acts by the identity, then it is obvious that $\mathbb{R}^n/G = \mathbb{R}^n$ is a smooth manifold.

Conversely, let us consider G-invariant homogeneous polynomials $p_1, \ldots, p_k$ generating the algebra of invariant polynomials: $\mathbb{R}[x_1, \ldots, x_n]^G = \mathbb{R}[p_1, \ldots, p_k]$. According to Theorem 11.14, the map $p = (p_1, \ldots, p_k) \colon \mathbb{R}^n \to \mathbb{R}^k$ induces an isomorphism $p\colon \mathbb{R}^n/G = Z$, where $Z := p(\mathbb{R}^k)$ is a closed subset in $\mathbb{R}^k$. Let us consider a G-invariant scalar product on $\mathbb{R}^n$ and let r^2 be the square of the distance to the origin of $\mathbb{R}^n$. Since r^2 is G–invariant, we have $r^2 = F(p_1,\ldots,p_k)$ for some $F \in \mathcal{C}^\infty(\mathbb{R}^n/G) = \mathcal{C}^\infty(Z)$. Since $F \geq 0$ on the smooth manifold Z and $F(\,\overline{0}\,) = 0$, the differential of F at $\overline{0}$ (origin in $\mathbb{R}^n/G = Z$) vanishes, i.e.,

$$F(x_1,\ldots,x_k) = \sum_{i,j=1}^{k} a_{ij}x_ix_j \quad , \qquad \text{in } \; \mathcal{C}^\infty(Z)/\mathfrak{m}_{\overline{0}}^3$$

where the coefficients a_{ij} are real numbers and $\mathfrak{m}_{\overline{0}} = (x_1,\ldots,x_k)$ stands for the ideal of all differentiable functions on $Z \subseteq \mathbb{R}^k$ vanishing at $\overline{0}$. Composing F with the map $p = (p_1,\ldots,p_k) : \mathbb{R}^n \to Z \subseteq \mathbb{R}^k$ we obtain

$$r^2 = F(p_1,\ldots,p_k) = \sum_{i,j=1}^{k} a_{ij}p_ip_j \quad , \qquad \text{in } \; \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{m}_0^3$$

Now, r^2 is an homogeneous polynomial function on $\mathbb{R}^n$ of degree 2, so that the homogeneous polynomials p_i and p_j have degree 1 whenever $a_{ij}p_ip_j \neq 0$. Hence r^2 is a polynomial in G-invariant linear forms, so that the G-invariant linear forms span the dual space $(\mathbb{R}^n)^*$ and we conclude that G acts on $\mathbb{R}^n$ by the identity.

□

Theorem 11.27. *Let $G \times \mathcal{V} \to \mathcal{V}$ be a proper differentiable action of a Lie group G on a paracompact smooth manifold $\mathcal{V}$. The following conditions are equivalent:*

(a) *The geometric quotient $\mathcal{V}/G$ is a smooth manifold.*

(b) *The isotropy subgroups are locally constant up to conjugation.*

Moreover, in such a case, the quotient map $\pi\colon \mathcal{V} \to \mathcal{V}/G$ is a submersion.

Proof. We use the notations of 11.24 and 11.25.

$(b) \Rightarrow (a)$. Let $\overline{x} = \pi(x) \in \mathcal{V}/G$ and let U be a neighbourhood of x in $\mathcal{V}$ where the isotropy type of the orbits is constant. We may assume that U is a G-invariant open neighbourhood of x, because $I_{gy} = gI_yg^{-1}$, and even that U is isomorphic to the normal vector bundle N of the orbit Gx. Since the map $N \to Gx$ is equivariant, it is obvious that $I_y \subseteq I_x$ for any $y \in N_x$; hence, I_y and I_x being conjugated compact subgroups of G, we conclude ([4] IX.9 Lemma 6) that $I_y = I_x$. Therefore I_x acts on N_x by the identity, so that $N/G = N_x/I_x = N_x$ is a smooth manifold. Now, $N/G \simeq U/G$ is a neighbourhood of $\overline{x}$ in X/G and we conclude that X/G is a smooth manifold.

$(a) \Rightarrow (b)$. Since the problem is local, we may assume that $\mathcal{V} \to \mathcal{V}/G$ coincides with $N \to N/G$. Since $N_x/I_x = N/G$ by 11.25, it results that N_x/I_x is a smooth manifold. By 11.26, we have that I_x acts by the identity on N_x. Recall that the isotropy subgroups of G corresponding to points of N_x are included in I_x, hence such subgroups coincide with I_x. Finally, N_x and N have common isotropy subgroups (up to conjugation) since $GN_x = N$.

Moreover, the natural inclusion $N/G = N_x/I_x = N_x \hookrightarrow N$ is a section of the quotient map $N \to N/G$. Therefore, the quotient map $\mathcal{V} \to \mathcal{V}/G$ is a submersion since it admits local sections.

□

Definition. Given a differentiable space X, a point $x \in X$ is said to be non-singular if there exists an open neighbourhood U of x in X which is a smooth manifold. In other case, x is said to be a **singular point** of X.

Let $G \times \mathcal{V} \to \mathcal{V}$ be a differentiable proper action of a Lie group G on a paracompact smooth manifold $\mathcal{V}$. Let us recall some well known facts about the the isotropy type of the points of $\mathcal{V}$ (see [4] IX.9, Th.2 and exer. 9).

The isotropy type has the following semicontinuity property: Given a point $x \in \mathcal{V}$, there exists an open neighbourhood U of x such that $\tau_u \leq \tau_x$ for any $u \in U$.

Given a type of isotropy τ, the subset

$$\mathcal{V}_\tau \ := \ \{x \in \mathcal{V} : \tau_x = \tau\}$$

is a locally closed submanifold of $\mathcal{V}$ and the stratification

$$\mathcal{V} \ = \ \coprod_\tau \mathcal{V}_\tau$$

is locally finite. This stratification is finite when $\mathcal{V}/G$ is compact or $\mathcal{V} = \mathbb{R}^n$ and the action is linear.

Moreover, if $\mathcal{V}$ is connected and τ_0 denotes the minimal isotropy type of the points of $\mathcal{V}$, then the stratum $\mathcal{V}_{\tau_0}$ is a dense open subset of $\mathcal{V}$.

Combining these facts with Theorem 11.27 we obtain directly the following result.

Theorem 11.28. *Let $G\times\mathcal{V} \to \mathcal{V}$ be a differentiable proper action of a Lie group G on a paracompact smooth manifold $\mathcal{V}$. We have:*

1. *The set of all non-singular points of $\mathcal{V}/G$ is a dense open subset.*
2. *There exists a locally finite stratification by locally closed subsets*

$$\mathcal{V}/G = \coprod_\tau \mathcal{V}_\tau/G$$

whose strata $\mathcal{V}_\tau/G$ are smooth manifolds.

11.6 Differentiable Groups

Lie groups are the group objects in the category of smooth manifolds. In the category of differentiable spaces the group objects are named differentiable groups:

Definition. A **differentiable group** is a differentiable space G endowed with morphisms $m\colon G\times G\to G$, $inv\colon G\to G$ and a fixed point $1\in G$, satisfying the following properties:

(i) The following diagram is commutative:

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{m\times Id} & G\times G \\ {\scriptstyle Id\times m}\downarrow & & {\scriptstyle m}\downarrow \\ G\times G & \xrightarrow{m} & G \end{array}$$

(ii) The morphism $m\colon 1\times G\to G$ is the identity.

(iii) The composition morphism

$$G\times G \xrightarrow{Id\times inv} G\times G \xrightarrow{m} G$$

coincides with the constant morphism $G\times G\longrightarrow\{1\}\subseteq G$.

In other words, a differentiable group is a differentiable space G whose functor of points $G^\bullet$ is valued in the category of abstract groups.

A differentiable group is said to be a **formal group** if the underlying set has a unique point.

The notions of morphism of differentiable groups and of differentiable action of a differentiable group on a differentiable space are defined in the obvious way.

Of course, any Lie group is a differentiable group.

The typical way to construct a differentiable group is the following one. Let H be a closed Lie subgroup of a Lie group G. Let $\mathbf{W}_H$ be the Whitney space corresponding to the closed subset $H\subseteq G$, then the Lie group structure of G induces a structure of differentiable group on $\mathbf{W}_H$. When $H=\{1\}$ is the trivial subgroup, then we obtain a formal group $\mathbf{W}_1$.

Differentiable groups are interesting in their own right. We limit ourselves to state without proofs some illustrative facts:

1. The category of finite-dimensional Lie $\mathbb{R}$-algebras is equivalent to the category of formal groups.
2. Let $\widehat{\mathbb{G}}_a=\operatorname{Spec}_r\mathbb{R}[[t]]$ be the Whitney space of the origin $\{0\}$ of the additive line $\mathbb{R}$. This formal group has the following property: Given a differentiable space X, there exists a natural bijection between the set of differentiable actions of $\widehat{\mathbb{G}}_a$ on X and the set of tangent vector fields ($\mathbb{R}$-linear derivations $D\colon\mathcal{O}_X\to\mathcal{O}_X$) on X.
3. Any differentiable group is a formally smooth differentiable space; hence, by 10.20 they are locally Whitney spaces (see [66]).

Appendix

A Sheaves of Fréchet Modules

This appendix is a natural complement of chapter 8, which was dedicated to the properties of Fréchet modules over a differentiable algebra. Let A be a differentiable algebra and let $X = \mathrm{Spec}_r A$. By the Localization theorem of Fréchet modules, we know that any Fréchet A-module M defines by localization a sheaf $\widetilde{M}$ on X,

$$\widetilde{M}(U) = M_U .$$

Moreover, M_U is a Fréchet A_U-module by theorem 8.9, that is to say, the sheaf $\widetilde{M}$ is valued in the category of Fréchet vector spaces. This kind of sheaf occurs naturally on differentiable spaces. For example, the structural sheaf $\mathcal{O}_X$, the sheaf of differentials Ω_X and the sheaf of r-jets J^r_X are Fréchet sheaves. The role of Fréchet sheaves in the theory of differentiable spaces is similar to that of quasi-coherent sheaves in algebraic geometry.

In this appendix, we shall introduce the notion of a Fréchet sheaf and some natural operations: tensor product, direct image and inverse image of Fréchet sheaves. The main result is the equivalence of Fréchet modules and Fréchet sheaves on affine differentiable spaces (theorem A.3).

Finally, we shall construct certain differentiable spaces associated to Fréchet sheaves (in particular, vector bundles associated to locally free $\mathcal{O}_X$-modules).

A.1 Sheaves of Locally Convex Spaces

Definition. Let $\mathcal{F}$ be a sheaf of real vector spaces on a topological space X. We say that $\mathcal{F}$ is a **sheaf of locally convex spaces** if, for any open set $U \subseteq X$, we have a structure of locally convex space on $\mathcal{F}(U)$ such that, for any open cover $\{U_i\}$ of U, the topology of $\mathcal{F}(U)$ is the initial topology for the restriction morphisms $\mathcal{F}(U) \to \mathcal{F}(U_i)$, i.e., the map $\mathcal{F}(U) \hookrightarrow \prod_i \mathcal{F}(U_i)$ is a topological immersion. In such a case we also say that we have a locally convex topology on the sheaf $\mathcal{F}$.

Note that the restriction morphisms $\mathcal{F}(U) \to \mathcal{F}(V)$ are continuous for any open subsets $V \subseteq U$ in X.

A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ between sheaves of locally convex spaces is said to be **continuous** if the corresponding map $\mathcal{F}(U) \to \mathcal{G}(U)$ is continuous for any open subset $U \subseteq X$.

J.A. Navarro González and J.B. Sancho de Salas: LNM 1824, pp. 151–179, 2003.

The concept of **sheaf of locally m-convex algebras** $\mathcal{O}_X$ is defined analogously. In such a case, any $\mathcal{O}_X$-module $\mathcal{M}$ endowed with a locally convex topology such that $\mathcal{M}(U)$ is a locally convex $\mathcal{O}_X(U)$-module for any open subset $U \subseteq X$, is said to be a **locally convex $\mathcal{O}_X$-module**. **Morphisms of locally convex $\mathcal{O}_X$-modules** $\mathcal{M} \to \mathcal{N}$ are defined to be continuous morphisms of $\mathcal{O}_X$-modules.

"Recollement" theorem. *Let $\mathcal{F}$ be a sheaf of real vector spaces on a topological space X and let $\{U_i\}$ be an open cover of X. If we have a locally convex topology on each sheaf $\mathcal{F}|_{U_i}$, and they coincide on the intersections $U_i \cap U_j$, then there exists a unique locally convex topology on $\mathcal{F}$ inducing on each sheaf $\mathcal{F}|_{U_i}$ the given topology.*

A similar result holds for sheaves of $\mathbb{R}$-algebras and $\mathcal{O}_X$-modules.

Proof. The uniqueness is obvious, since the topology of $\mathcal{F}(U)$ must be the initial topology of the restriction morphisms $\mathcal{F}(U) \to \mathcal{F}(U_i \cap U) = \mathcal{F}|_{U_i}(U_i \cap U)$. To prove the existence, let us consider such topology on $\mathcal{F}(U)$. The hypotheses imply that if U is contained in some U_i, then the bijection $\mathcal{F}(U) = \mathcal{F}|_{U_i}(U)$ is a homeomorphism. Finally, we have to prove that $\mathcal{F}(U) \hookrightarrow \prod_j \mathcal{F}(V_j)$ is a topological immersion for any open cover $\{V_j\}$ of U. Let us consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \prod_i \mathcal{F}(U_i \cap U) = \prod_i \mathcal{F}|_{U_i}(U_i \cap U) \\ \downarrow & & \downarrow \\ \prod_j \mathcal{F}(V_j) & \longrightarrow & \prod_{ij} \mathcal{F}(U_i \cap V_j) = \prod_{ij} \mathcal{F}|_{U_i}(U_i \cap V_j) \end{array}$$

The top and down arrows are topological immersions by definition of the topology on $\mathcal{F}$. The right arrow is a topological immersion because $\{U_i \cap V_j\}_j$ is an open cover of $U_i \cap U$ and each $\mathcal{F}|_{U_i}$ is a sheaf of locally convex spaces. Now the diagram shows that the left arrow also is a topological immersion.

□

Corollary A.1. *If $(X, \mathcal{O}_X)$ is a differentiable space, then the sheaf of algebras $\mathcal{O}_X$ admits a unique locally m-convex topology such that, for any affine open subspace V, the topology induced on the differentiable algebra $\mathcal{O}_X(V)$ coincides with the canonical topology.*

Proof. By the above Recollement theorem, we may assume that X is affine, $X = \text{Spec}_r A$. The uniqueness is obvious. To prove the existence, it is enough to show that the canonical Fréchet topology of $\mathcal{O}_X(U) = A_U$ is the initial topology of the restriction morphisms $\mathcal{O}_X(U) = A_U \to A_{U_i} = \mathcal{O}_X(U_i)$, whenever U is an open set in X and $\{U_i\}$ is an open cover of U. Since the localization morphisms are continuous, the canonical topology is finer than the initial topology. Moreover, the following exact sequence

$$\mathcal{O}_X(U) \longrightarrow \prod_i \mathcal{O}_X(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)$$

let us deduce that the initial topology of the morphisms $\mathcal{O}_X(U) \to \mathcal{O}_X(U_i)$ also is Fréchet, hence both topologies coincide.

□

Definition. Let $(X, \mathcal{O}_X)$ be a differentiable space. A locally convex $\mathcal{O}_X$-module $\mathcal{M}$ is said to be **Fréchet** when $\mathcal{M}(V)$ is Fréchet for any affine open subspace V of X.

Of course, morphisms of Fréchet $\mathcal{O}_X$-modules $\mathcal{M} \to \mathcal{N}$ are morphism of locally convex $\mathcal{O}_X$-modules, that is to say, continuous morphisms of $\mathcal{O}_X$-modules.

A **Fréchet $\mathcal{O}_X$-algebra** is a sheaf $\mathcal{A}$ of locally m-convex algebras endowed with a continuous morphism $\mathcal{O}_X \to \mathcal{A}$ of sheaves of algebras, such that $\mathcal{A}(V)$ is a Fréchet algebra for any affine open subspace V of X. Morphisms of Fréchet $\mathcal{O}_X$-algebras $\mathcal{A} \to \mathcal{B}$ are continuous morphisms of $\mathcal{O}_X$-algebras.

Lemma A.2. *Let $\{U_i\}$ be an open cover of a differentiable space $(X, \mathcal{O}_X)$ and let $\mathcal{M}$ be a locally convex $\mathcal{O}_X$-module. $\mathcal{M}$ is Fréchet if and only if so is $\mathcal{M}|_{U_i}$ for any index i.*

Proof. It is obvious that $\mathcal{M}|_{U_i}$ is Fréchet whenever $\mathcal{M}$ is Fréchet. Conversely, if the sheaves $\mathcal{M}|_{U_i}$ are Fréchet, and U is an affine open subspace of X, then $\{U_i \cap U\}$ has a countable subcover $\{U_n \cap U\}$. Since $\mathcal{M}(U_n \cap U)$ and $\mathcal{M}(U_n \cap U_m \cap U)$ are Fréchet, the following exact sequence

$$\mathcal{M}(U) \longrightarrow \prod_n \mathcal{M}(U_n \cap U) \rightrightarrows \prod_{n,m} \mathcal{M}(U_n \cap U_m \cap U)$$

let us conclude that $\mathcal{M}(U)$ es Fréchet.

□

Theorem A.3. *Let A be a differentiable algebra and let $X = \operatorname{Spec}_r A$. The covariant functors $M \rightsquigarrow \tilde{M}$ and $\mathcal{M} \rightsquigarrow \Gamma(X, \mathcal{M})$ define an equivalence of the category of Fréchet A-modules with the category of Fréchet $\tilde{A}$-modules.*

Proof. If M is a Fréchet A-module, then $\tilde{M}(U) = M_U$ for any open set U in X, by the Localization theorem for Fréchet A-modules. In particular, we have $M = \Gamma(X, \tilde{M})$. The same argument given in the proof of A.2 shows that $\tilde{M}$ is a Fréchet $\tilde{A}$-module when the A_U-module $\tilde{M}(U) = M_U$ is endowed with the localization topology (which is Fréchet by 8.9).

Conversely, if $\mathcal{M}$ is a Fréchet $\tilde{A}$-module and we put $M := \Gamma(X, \mathcal{M})$, then the natural morphism $M_U \to \mathcal{M}(U)$ is an algebraic isomorphism by 3.11. It is continuous because of the universal property of the localization M_U, and it is a homeomorphism since both topologies are Fréchet. The topology of $\mathcal{M}(U)$ by hypothesis, and the topology of M_U by 8.9.

□

Definition. Let $(X, \mathcal{O}_X)$ be a differentiable space. A submodule $\mathcal{N}$ of a locally convex $\mathcal{O}_X$-module $\mathcal{M}$ is said to be **closed** when $\mathcal{N}(V)$ is closed in $\mathcal{M}(V)$ for any affine open subspace V of X (hence for any open subspace V of X).

Proposition A.4. *Let $(X, \mathcal{O}_X)$ be a differentiable space and let $\mathcal{M}$ be a Fréchet $\mathcal{O}_X$-module. If $\mathcal{N}$ is a submodule of $\mathcal{M}$, then there exists a unique closed submodule $\overline{\mathcal{N}}$ such that*

$$\overline{\mathcal{N}}(V) = \overline{\mathcal{N}(V)}$$

for any affine open subspace V of X.

Furthermore, there exists a unique structure of Fréchet $\mathcal{O}_X$-module on $\mathcal{M}/\overline{\mathcal{N}}$ such that the canonical projection $\mathcal{M} \to \mathcal{M}/\overline{\mathcal{N}}$ is a continuous morphism.

Proof. First we assume that X is affine. Let $N = \mathcal{N}(X)$ and $M = \mathcal{M}(X)$, hence $\mathcal{N}(U) = N_U$ and $\mathcal{M}(U) = M_U$ for any open subset $U \subseteq X$ by 3.11.

The uniqueness of the submodule $\overline{\mathcal{N}}$ is obvious. To show that it exists, we have to prove that $\overline{\mathcal{N}}$ is the associated sheaf of the Fréchet A-module $\overline{N}$, that is to say, $\overline{(N_U)} = \overline{N}_U$ for any open set $U \subseteq X$. The restriction morphism $M \to M_U$ is continuous and it transforms N into N_U; hence $\overline{N}$ into $\overline{(N_U)}$ and it follows that $\overline{N}_U \subseteq \overline{(N_U)}$. Conversely, since $N \subseteq \overline{N}$, we have that $N_U \subseteq \overline{N}_U$ and, $\overline{N}_U$ being a closed submodule of M_U by 8.10, we conclude that $\overline{(N_U)} \subseteq \overline{N}_U$.

Moreover, it is clear that there exists a unique Fréchet topology on $M/\overline{N}$ such that the canonical projection $M \to M/\overline{N}$ is continuous, so that A.3 let us conclude the existence of a unique structure of Fréchet $\mathcal{O}_X$-module on $\mathcal{M}/\overline{\mathcal{N}}$ such that the canonical projection $\mathcal{M} \to \mathcal{M}/\overline{\mathcal{N}}$ is continuous.

In the general case, the uniqueness follows directly from the affine case and the Recollement theorem, while the existence follows from the affine case, the uniqueness and the Recollement theorem.

□

A.2 Examples

Example. Let X be a differentiable space. The $\mathcal{O}_X$-module $\mathcal{O}_X^r$ admits a unique structure of Fréchet $\mathcal{O}_X$-module.

If A is a Fréchet algebra, then the natural topology of the direct sum A^r is finer than any other structure of locally convex A-module, hence it is the unique structure of Fréchet A-module on A^r. Now the affine case is a consequence of A.3, and the general case follows from the affine case, the uniqueness and the Recollement theorem.

Example. Let X be a differentiable space. Any locally free $\mathcal{O}_X$-module of finite rank $\mathcal{E}$ admits a unique structure of Fréchet $\mathcal{O}_X$-module.

If $\mathcal{E}$ is trivial, $\mathcal{E} \simeq \mathcal{O}_X^r$, then it admits a unique Fréchet topology, according to the former example. The general case follows from the trivial case, the uniqueness and the Recollement theorem.

Example. Let X be a differentiable space. If a locally finitely generated $\mathcal{O}_X$-module $\mathcal{M}$ admits a structure of Fréchet $\mathcal{O}_X$-module, then this structure is unique. In such a case, any morphism of $\mathcal{O}_X$-modules $\mathcal{M} \to \mathcal{N}$ into a locally convex $\mathcal{O}_X$-module $\mathcal{N}$ is continuous.

By the Recollement theorem, we may assume that X is affine, $X = \mathrm{Spec}_r\, A$, and that $\mathcal{M}$ is the sheaf associated to some finitely generated A-module M. In such a case we fix an epimorphism $A^r \to M$. It is continuous with respect to any structure of locally convex A-module on the A-module M, so that the final topology of this epimorphism is the only possible Fréchet structure, and it exists if and only if the kernel is a closed submodule. In such a case, any morphism of A-modules $M \to N$ into a locally convex A-module N is continuous, since so is any morphism of A-modules $A^r \to N$ and we consider on M the quotient topology induced by A^r.

Example (Module of relative differentials). Let $\varphi \colon X \to S$ be a morphism of differentiable spaces. Let $\mathcal{D}_{X/S}$ be the sheaf of ideals of the diagonal embedding $X \hookrightarrow X \times_S X$ (rigorously, it is a sheaf of closed ideals on an open neighbourhood of the diagonal subspace). Using A.4, we may re-define the sheaf of relative differentials as a Fréchet $\mathcal{O}_X$-module:

$$\Omega_{X/S} \;=\; \mathcal{D}_{X/S}/\overline{\mathcal{D}^2_{X/S}}$$

(the $\mathcal{O}_X$-module structure is defined by the second projection $p_2 \colon X \times_S X \to X$). Moreover, this structure of Fréchet $\mathcal{O}_X$-module is unique since $\Omega_{X/S}$ is a locally finitely generated $\mathcal{O}_X$-module by 10.12.

Example (Tensor products). Let $\mathcal{M}$ and $\mathcal{N}$ be Fréchet $\mathcal{O}_X$-modules on a differentiable space X. Let us define the tensor product sheaf $\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{N}$.

Firstly, let us assume that X is affine: $X = \mathrm{Spec}_r\, A$. Then $\mathcal{M} = \widetilde{M}$ and $\mathcal{N} = \widetilde{N}$, where $M := \Gamma(X, \widetilde{M})$ and $N := \Gamma(X, \widetilde{N})$ are Fréchet A-modules. Now the Fréchet A-module $M\widehat{\otimes}_A N$ defines, by localization, a Fréchet $\mathcal{O}_X$-module:

$$\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{N} \;:=\; (M\widehat{\otimes}_A N)^{\sim} \ .$$

For any open set U in X, we have

$$(\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{N})(U) \;=\; (M\widehat{\otimes}_A N)_U \;=\; M_U\widehat{\otimes}_{A_U} N_U \;=\; \mathcal{M}(U)\widehat{\otimes}_{\mathcal{O}_X(U)}\mathcal{N}(U) \ ,$$

where the second equality is obtained in the following way:

$$\begin{aligned}(M\widehat{\otimes}_A N)_U \;&\overset{8.11}{=}\; (M\widehat{\otimes}_A N)\widehat{\otimes}_A A_U \;\overset{6.10}{=}\; (M\widehat{\otimes}_A A_U)\widehat{\otimes}_{A_U}(N\widehat{\otimes}_A A_U) \;= \\ &=\; M_U\widehat{\otimes}_{A_U} N_U \ .\end{aligned}$$

These equalities $(\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{N})(U) = \mathcal{M}(U)\widehat{\otimes}_{\mathcal{O}_X(U)}\mathcal{N}(U)$ imply that

$$(\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{N})|_V \;=\; (\mathcal{M}|_V)\widehat{\otimes}_{\mathcal{O}_V}(\mathcal{N}|_V)$$

for any open set V in X. This fact let us define the sheaf $\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{N}$ by "recollement" when X is not affine: Given Fréchet $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$ on a differentiable space X, the sheaf $\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{N}$ is defined to be the unique Fréchet $\mathcal{O}_X$-module such that

$$(\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{N})|U \;=\; (\mathcal{M}|_U)\widehat{\otimes}_{\mathcal{O}_U}(\mathcal{N}|_U)$$

for any affine open subset $U \subseteq X$.

Example (Direct images). Let $\varphi\colon Y \to X$ be a morphism of differentiable spaces. If $\mathcal{N}$ is a locally convex $\mathcal{O}_Y$-module, then $\varphi_*\mathcal{N}$ is a locally convex $\mathcal{O}_X$-module. In fact, $(\varphi_*\mathcal{N})(U) := \mathcal{N}(\varphi^{-1}U)$ is a locally convex $\mathcal{O}_X(U)$-module because $\mathcal{N}(\varphi^{-1}U)$ is a locally convex $\mathcal{O}_Y(\varphi^{-1}U)$-module and the morphism $\varphi^*\colon \mathcal{O}_X(U) \to \mathcal{O}_Y(\varphi^{-1}U)$ is continuous, as we may see considering an affine open cover of $\varphi^{-1}U$ and applying 2.23.

In general, it may be false that $\varphi_*\mathcal{N}$ is Fréchet when so is $\mathcal{N}$ (consider the projection of an uncountable discrete space over a point). However, if φ is an **affine morphism**, in the sense that $\varphi^{-1}(U)$ is affine for any affine open subspace $U \subseteq X$, then $\varphi_*\mathcal{N}$ is a Fréchet $\mathcal{O}_X$-module whenever $\mathcal{N}$ is a Fréchet $\mathcal{O}_Y$-module. In fact, by hypothesis $(\varphi_*\mathcal{N})(U) = \mathcal{N}(\varphi^{-1}U)$ is a Fréchet $\mathcal{O}_Y(\varphi^{-1}U)$-module, hence it is a Fréchet $\mathcal{O}_X(U)$-module, for any affine open subspace $U \subseteq X$.

In the affine case, $X = \operatorname{Spec}_r A$ and $Y = \operatorname{Spec}_r B$, we have that $\mathcal{N}$ is the sheaf associated to a Fréchet B-module N and, by 8.12, $\varphi_*\mathcal{N}$ is just the sheaf associated to the Fréchet A-module N.

A.3 Inverse Image

Theorem A.5. *Let $\varphi\colon Y \to X$ be a morphism of differentiable spaces and let $\mathcal{M}$ be a Fréchet $\mathcal{O}_X$-module. There exists a Fréchet $\mathcal{O}_Y$-module $\varphi^*\mathcal{M}$ such that, for any Fréchet $\mathcal{O}_Y$-module $\mathcal{N}$, continuous morphisms of $\mathcal{O}_Y$-modules $\varphi^*\mathcal{M} \to \mathcal{N}$ correspond with continuous morphisms of $\mathcal{O}_X$-modules $\mathcal{M} \to \varphi_*\mathcal{N}$:*

$$\operatorname{Hom}_{\mathcal{O}_Y}(\varphi^*\mathcal{M}, \mathcal{N}) \;=\; \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \varphi_*\mathcal{N}) \;.$$

Proof. The theorem holds in the affine case, $Y = \operatorname{Spec}_r B$ and $X = \operatorname{Spec}_r A$, because $\mathcal{M}$ is the sheaf associated to some Fréchet A-module M and the sheaf associated to the Fréchet B-module $M\widehat{\otimes}_A B$ has the required universal property by A.3 and the universal property of the base change of locally convex modules:

$$\operatorname{Hom}_B(M\widehat{\otimes}_A B, N) \;=\; \operatorname{Hom}_A(M, N) \;.$$

The theorem holds for any open embedding $j\colon U \hookrightarrow X$ (just put $j^*\mathcal{M} := \mathcal{M}|_U$) and, whenever it holds for two morphisms $V \xrightarrow{\varphi} U \xrightarrow{j} X$, then it also holds for the composition $j\varphi\colon V \to X$, since we may take $(j\varphi)^*\mathcal{M} := \varphi^*(j^*\mathcal{M})$. Therefore, if the image $\varphi(V_i)$ of an affine open subspace of Y is contained in some affine open subspace of X, then the theorem holds for the restriction $\varphi_i\colon V_i \to X$ of φ. We may consider an open cover $\{V_i\}$ of Y by such affine open subspaces since they define a basis of the topology of Y. Since the representant of a functor is unique, the Recollement theorem let us obtain the existence of a locally convex $\mathcal{O}_Y$-module $\varphi^*\mathcal{M}$ such that $\varphi^*\mathcal{M}|_{V_i} = \varphi_i^*\mathcal{M}$ for any index i, and it is Fréchet by A.2. The following exact sequences (where we put $V_{ij} = V_i \cap V_j$, $(\varphi^*\mathcal{M})_i = (\varphi^*\mathcal{M})|_{V_i}$, $(\varphi^*\mathcal{M})_{ij} = (\varphi^*\mathcal{M})|_{V_{ij}}$, $\mathcal{N}_i = \mathcal{N}|_{V_i}$ and $\mathcal{N}_{ij} = \mathcal{N}|_{V_{ij}}$)

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{O}_Y}(\varphi^*\mathcal{M},\mathcal{N}) & \to & \prod_i \mathrm{Hom}_{\mathcal{O}_Y}((\varphi^*\mathcal{M})_i,\mathcal{N}_i) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{\mathcal{O}_Y}((\varphi^*\mathcal{M})_{ij},\mathcal{N}_{ij}) \\
 & & \| & & \| \\
 & & \prod_i \mathrm{Hom}_{\mathcal{O}_Y}(\varphi_i^*\mathcal{M},\mathcal{N}_i) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{\mathcal{O}_Y}(\varphi_{ij}^*\mathcal{M},\mathcal{N}_{ij}) \\
 & & \| & & \| \\
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\varphi_*\mathcal{N}) & \to & \prod_i \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\varphi_{i*}(\mathcal{N}_i)) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\varphi_{ij*}(\mathcal{N}_{ij}))
\end{array}$$

let us conclude that $\mathrm{Hom}_{\mathcal{O}_Y}(\varphi^*\mathcal{M},\mathcal{N}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\varphi_*\mathcal{N})$.

□

Definition. We say that $\varphi^*\mathcal{M}$ is the **inverse image** of the Fréchet $\mathcal{O}_X$-module $\mathcal{M}$ by the morphism φ, and we also denote it by $\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{O}_Y$.

Remark A.6. The inverse image is transitive: $\phi^*(\varphi^*\mathcal{M}) = (\varphi\phi)^*\mathcal{M}$.

Remark A.7. If $j\colon U \hookrightarrow X$ is an open embedding, then $j^*\mathcal{M} = \mathcal{M}|_U$.

Remark A.8. If $\varphi\colon \mathrm{Spec}_r B \to \mathrm{Spec}_r A$ is a morphism of affine differentiable spaces and M is a Fréchet A-module, then $\varphi^*(\widetilde{M})$ is the sheaf associated to the Fréchet B-module $M\widehat{\otimes}_A B$.

Remark A.9. The inverse image of Fréchet modules preserves epimorphisms.

Proof. Any surjective continuous morphism of A-modules $M \to M'$ between Fréchet A-modules is an open map, hence the morphism $M\widehat{\otimes}_A B \to M'\widehat{\otimes}_A B$ is surjective (see 6.5).

□

Remark A.10. If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of finite rank, then

$$\varphi^*\mathcal{E} = \mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_Y \ .$$

Proof. It is enough to consider the affine case, $Y = \mathrm{Spec}_r B$ and $X = \mathrm{Spec}_r A$, and we may assume that $\mathcal{E}$ is the sheaf associated to A^r. This case follows from the equalities $(A^r)\widehat{\otimes}_A B = \widehat{B}^r = B^r = (A^r)\otimes_A B$ (see 6.7 and 6.8).

□

Remark A.11. If $\mathcal{I}$ is a closed ideal of $\mathcal{O}_X$, then we have

$$(\mathcal{O}_X/\mathcal{I})\widehat{\otimes}_{\mathcal{O}_X}\mathcal{O}_Y = \mathcal{O}_Y/\overline{\mathcal{I}\mathcal{O}_Y} \ .$$

Proof. 6.8.

□

Given a morphism $X \to S$ of differentiable spaces, let us recall that $\Omega_{X/S}$ denotes the Fréchet sheaf of relative differentials. Now we shall reformulate some results of chapter 10 in terms of sheaves of relative differentials.

Proposition A.12. *If* $p_1\colon X\times_S T\to X$ *denotes the first projection, then*

$$\Omega_{X\times_S T/T} \;=\; p_1^*\Omega_{X/S}\;.$$

Proof. 10.8.

□

Proposition A.13. $\Omega_{X\times_S Y/S} \;=\; p_1^*\Omega_{X/S}\oplus p_2^*\Omega_{Y/S}\;.$

Proof. 10.9.

□

Proposition A.14. *Let* $\varphi\colon X\to Y$ *be a morphism of differentiable* S*-spaces. If the natural morphism* $\varphi^*\Omega_{Y/S}\to\Omega_{X/S}$ *admits a continuous retraction, then we have an exact sequence:*

$$0\;\longrightarrow\;\varphi^*\Omega_{Y/S}\;\longrightarrow\;\Omega_{X/S}\;\longrightarrow\;\Omega_{X/Y}\;\longrightarrow\;0\;.$$

Proof. The first sequence of differentials.

□

Proposition A.15. *If* $\varphi\colon X\to Y$ *is a smooth morphism between differentiable* S*-spaces, then we have an exact sequence:*

$$0\;\longrightarrow\;\varphi^*\Omega_{Y/S}\;\longrightarrow\;\Omega_{X/S}\;\longrightarrow\;\Omega_{X/Y}\;\longrightarrow\;0\;.$$

Proof. When $\varphi\colon X=U\times Y\to Y$ is the second projection, the natural morphism $\varphi^*\Omega_{Y/S}\to\Omega_{X/S}$ admits the continuous retraction $\Omega_{X/S}\to\Omega_{U\times Y/U\times S}=\varphi^*\Omega_{Y/S}$.

□

Proposition A.16. *If* $\varphi\colon X\to Y$ *is a smooth morphism between smooth differentiable* S*-spaces, then we have an exact sequence:*

$$0\;\longrightarrow\;\Omega_{Y/S}\otimes_{\mathcal{O}_Y}\mathcal{O}_X\;\longrightarrow\;\Omega_{X/S}\;\longrightarrow\;\Omega_{X/Y}\;\longrightarrow\;0\;.$$

Proof. Since Y is smooth over S, we have that $\Omega_{Y/S}$ is a locally free $\mathcal{O}_Y$-module of finite rank, hence $\Omega_{Y/S}\widehat{\otimes}_{\mathcal{O}_Y}\mathcal{O}_X=\Omega_{Y/S}\otimes_{\mathcal{O}_Y}\mathcal{O}_X$ by A.10.

□

Proposition A.17. *Let* X *be a differentiable* S*-space and let* $\mathcal{I}$ *be the closed ideal of* $\mathcal{O}_X$ *defined by a closed embedding* $j\colon Y\hookrightarrow X$*. If the natural morphism* $\mathcal{I}/\overline{\mathcal{I}^2}\to j^*\Omega_{X/S}$ *admits a continuous retraction, then we have an exact sequence*

$$0\;\longrightarrow\;\mathcal{I}/\overline{\mathcal{I}^2}\;\longrightarrow\;j^*\Omega_{X/S}\;\longrightarrow\;\Omega_{Y/S}\;\longrightarrow\;0\;.$$

Proof. The second sequence of differentials.

□

A.4 Vector Bundles

Given two Fréchet $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{E}$ on a differentiable space X, we shall construct a differentiable X-space $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{E})$. As particular cases, we shall obtain the vector bundle associated to a locally free $\mathcal{O}_X$-module and the spaces of jets of morphisms (in appendix B).

Let S be a differentiable space and let F be a sheaf of sets on the category of differentiable S-spaces. Given an open set U in S, we will denote by F_U the restriction of F to the category of differentiable U-spaces. Note that F_U may be considered as a subsheaf of F, with the convention $F(X)=\emptyset$ whenever the structural morphism $X\to S$ does not factor through U.

Proposition A.18. *Let F be a sheaf of sets on the category of differentiable S-spaces. If there exists an open cover $\{U_i\}$ of S such that F_{U_i} is representable for any index i, then F is representable.*

Proof. Theorem 7.2, with the same proof, holds in the category of differentiable S-spaces. Now, given a differentiable S-space $T\to S$, let us denote $V_i=U_i\times_S T$. It is clear that $\{V_i\}$ is an open cover of T. For any morphism of functors $T^\bullet\to F$, it is easy to check that $F_{U_i}\times_F T^\bullet = V_i^\bullet$, hence $\{F_{U_i}\}$ is an open cover of F and, applying 7.2, we conclude that F is representable.

□

Lemma A.19. *If $\varphi\colon T\to \operatorname{Spec}_r A$ is a morphism of differentiable spaces, then the kernel of $\varphi^*\colon A\to\mathcal{O}_T(T)$ is a closed ideal of A.*

Proof. Let $\{U_i\}$ be a cover of T by affine open subsets. The kernel of each morphism $A\xrightarrow{\varphi^*}\mathcal{O}_T(T)\to\mathcal{O}_T(U_i)$ is a closed ideal of A by 2.23, and their intersection is just the kernel of $\varphi^*\colon A\to\mathcal{O}_T(T)$; hence it is closed.

□

Notation. Let X be a differentiable space and let $\mathcal{M}$ be a Fréchet $\mathcal{O}_X$-module. When a morphism of differentiable spaces $p\colon T\to X$ is considered as a parametrized point of X, then the inverse image $p^*\mathcal{M}$ is denoted by $\mathcal{M}|_p$ or $\mathcal{M}|_T$, and it is said to be the **restriction** of $\mathcal{M}$ to the point p. Analogously, the restriction of a morphism of Fréchet $\mathcal{O}_X$-modules $f\colon\mathcal{M}\to\mathcal{N}$ to the point p is defined to be the morphism of $\mathcal{O}_T$-modules

$$f|_p\colon \mathcal{M}|_p=\mathcal{M}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{O}_T\xrightarrow{f\otimes 1}\mathcal{N}\widehat{\otimes}_{\mathcal{O}_X}\mathcal{O}_T=\mathcal{N}|_p\ .$$

Theorem A.20. *Let X be a differentiable space and let $\mathcal{M}$, $\mathcal{E}$ be locally finitely generated Fréchet $\mathcal{O}_X$-modules. Let us consider the following contravariant functor defined on the category of differentiable X-spaces:*

$$F(T)\ =\ \operatorname{Hom}_{\mathcal{O}_T}(\mathcal{M}|_p,\mathcal{E}|_p)\ ,$$

for any parametrized point $p\colon T\to X$. If $\mathcal{E}$ is locally free, then F is representable by a differentiable X-space $\mathbf{Hom}(\mathcal{M},\mathcal{E})$, that is to say,

$$\operatorname{Hom}_X(T,\mathbf{Hom}(\mathcal{M},\mathcal{E}))\ =\ \operatorname{Hom}_{\mathcal{O}_T}(p^*\mathcal{M},p^*\mathcal{E})\ .$$

Proof. Note that F is a sheaf by A.7. The representability of F is a local problem on X by A.18; hence we may assume that X is affine, $X = \mathrm{Spec}_r\, A$, that $\mathcal{E} = \widetilde{A}^r$ and that $\mathcal{M}$ is the sheaf associated to a finitely generated Fréchet A-module $M = A^m/N$. We may also assume that $r = 1$, because if $\mathbf{Hom}(\mathcal{M}, \mathcal{O}_X)$ exists, then $\mathbf{Hom}(\mathcal{M}, \mathcal{O}_X^r) = \mathbf{Hom}(\mathcal{M}, \mathcal{O}_X) \times_X \overset{r}{\ldots} \times_X \mathbf{Hom}(\mathcal{M}, \mathcal{O}_X)$ also exists. In such a case we have

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_T}(p^*\mathcal{M}, p^*\mathcal{E}) = \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T^m/\overline{\mathcal{O}_T N}, \mathcal{O}_T) & \hookrightarrow & \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T^m, \mathcal{O}_T) \\ \| & & \| \\ \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T^m/\mathcal{O}_T N, \mathcal{O}_T) & & \mathrm{Hom}_X(T, \mathbb{R}^m \times X) \end{array}$$

(for the central equality recall that any morphism $\mathcal{O}_T^m \to \mathcal{O}_T$ is continuous). Therefore, morphisms of $\mathcal{O}_T$-modules $p^*\mathcal{M} \to p^*\mathcal{E}$ correspond with X-morphisms $(x_1(t), \ldots, x_m(t))\colon T \to \mathbb{R}^m \times X$ such that

$$x_1(t) f_{i1}(x) + \ldots + x_m(t) f_{im}(x) \ = \ 0 \ ,$$

where $n_i = (f_{i1}, \ldots, f_{im})$ runs over all the elements of $N \subseteq A^m$; hence, by 3.18, with morphisms of $\mathbb{R}$-algebras $\mathcal{O}_{\mathbb{R}^m \times X}(\mathbb{R}^m \times X) \to \mathcal{O}_T(T)$ which vanish on the ideal generated by the differentiable functions $x_1 f_{i1} + \ldots + x_m f_{im}$. By A.19 these morphisms vanish on the closure $\mathfrak{a}$ of such ideal, and we conclude that F is represented by the differentiable X-space $\mathrm{Spec}_r\, (\mathcal{O}_{\mathbb{R}^m \times X}(\mathbb{R}^m \times X)/\mathfrak{a})$.

□

With the previous notations, let $q\colon \mathbf{Hom}(\mathcal{M}, \mathcal{E}) \longrightarrow X$ be the structural morphism. According to Yoneda's lemma 7.1, the isomorphism of functors $\mathbf{Hom}(\mathcal{M}, \mathcal{E})^\bullet = F$ is determined by an element $\xi \in F(\mathbf{Hom}(\mathcal{M}, \mathcal{E}))$. This element is a morphism

$$\xi : \ q^*\mathcal{M} \longrightarrow q^*\mathcal{E}$$

called the **universal morphism**. Moreover, Yoneda's lemma let us describe the given isomorphism of functors in terms of the universal morphism in the following way: For any parametrized point $p\colon T \to X$, we have a bijection

$$\begin{array}{rcl} \mathrm{Hom}_X(T, \mathbf{Hom}(\mathcal{M}, \mathcal{E})) & = & \mathrm{Hom}_{\mathcal{O}_T}(p^*\mathcal{M}, p^*\mathcal{E}) \\ \lambda & \mapsto & \xi|_\lambda \end{array}$$

Definition. Let X be a differentiable space and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. The X-space $\mathbb{E} = \mathbf{Hom}(\mathcal{O}_X, \mathcal{E})$ is called the **vector bundle** associated to $\mathcal{E}$. According to A.20, we have a bijection

$$\mathrm{Hom}_X(T, \mathbb{E}) \ = \ \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T, x^*\mathcal{E}) \ = \ \Gamma(T, x^*\mathcal{E})$$

for any parametrized point $x\colon T \to X$. In particular, differentiable sections of the structural morphism $q\colon \mathbb{E} \to X$ correspond with sections of the sheaf $\mathcal{E}$,

$$\mathrm{Hom}_X(U, \mathbb{E}) \ = \ \Gamma(U, \mathcal{E})$$

for any open subset $U \hookrightarrow X$.

It is routine to prove, using the representability property of $\mathbf{Hom}(\mathcal{O}_X, \mathcal{E})$, the following properties:

–. For any morphism $\varphi\colon Y \to X$ of differentiable spaces, $\varphi^*\mathbb{E} := \mathbb{E} \times_X Y$ is the vector bundle associated to $\varphi^*\mathcal{E}$.

–. If $\mathcal{E} = \mathcal{O}_X \oplus \overset{n}{\dots} \oplus \mathcal{O}_X$ is a free $\mathcal{O}_X$-module, then the associated vector bundle is the trivial one: $\mathbb{E} = X \times \mathbb{R}^n \to X$.

Therefore, the vector bundle $\mathbb{E} \to X$ associated to any locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank n is locally isomorphic to the trivial one. As a consequence, if X is formally smooth then so is $\mathbb{E}$.

Proposition A.21. *Let X be a differentiable space, let $\mathcal{M}$ be a Fréchet $\mathcal{O}_X$-module and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. Given a morphism of $\mathcal{O}_X$-modules $f\colon \mathcal{M} \to \mathcal{E}$, there exists a closed differentiable subspace $Y \hookrightarrow X$ whose parametrized points are just points $p\colon T \to X$ such that the morphism $f|_p\colon \mathcal{M}|_p \to \mathcal{E}|_p$ vanishes.*

Proof. Note that the functor $F(T) = \{p \in X^\bullet(T)\colon\ f|_p = 0\}$ is a sheaf on the category of differentiable X-spaces. Since the representability of F is a local problem on X (see A.18), we may assume that X is affine, $X = \mathrm{Spec}_r\, A$, that $\mathcal{E} = \widetilde{A^r}$, and that $\mathcal{M}$ is the sheaf associated to a Fréchet A-module M. Now $f|_p = 0$ if and only if the morphism $p^*\colon A \to \mathcal{O}_T(T)$ vanishes on the ideal of A generated by the components of the elements of $f(M) \subseteq A^r$. Hence, by A.19, if and only if p^* vanishes on the closure $\mathfrak{a}$ of such ideal, and we conclude that the functor F is represented by the closed embedding $\mathrm{Spec}_r\,(A/\mathfrak{a}) \hookrightarrow \mathrm{Spec}_r\, A$.

□

Theorem A.22. *Let X be a differentiable space, and let $\mathcal{A}$, $\mathcal{B}$ be two Fréchet $\mathcal{O}_X$-algebras. Let us consider the following contravariant functor on the category of differentiable X-spaces:*

$$F(T) \ :=\ \mathrm{Hom}_{\mathcal{O}_T\text{-alg}}(\mathcal{A}|_p, \mathcal{B}|_p)$$

for any X-space $p\colon T \to X$. If $\mathcal{B}$ is locally free $\mathcal{O}_X$-module of finite rank, then F is representable by a differentiable X-space $\mathbf{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{A}, \mathcal{B})$, that is to say,

$$\mathrm{Hom}_X(T, \mathbf{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{A}, \mathcal{B})) \ =\ \mathrm{Hom}_{\mathcal{O}_T\text{-alg}}(\mathcal{A}|_p, \mathcal{B}|_p)\ .$$

Proof. Let us consider $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{O}_X$-modules. By A.20, there exists the differentiable X-space $q\colon \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B}) \longrightarrow X$. Let

$$\xi \,:\ q^*\mathcal{A} \longrightarrow q^*\mathcal{B}$$

be the universal morphism. Then $\mathbf{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{A}, \mathcal{B})$ is the differentiable subspace of $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B})$ of all (parametrized) points $\lambda\colon T \to \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B})$ such that

$$\xi|_\lambda \,:\ (q^*\mathcal{A})|_\lambda = \mathcal{A}|_p \longrightarrow (q^*\mathcal{B})|_\lambda = \mathcal{B}|_p$$

is a morphism of $\mathcal{O}_T$-algebras (for any X-space $p\colon T \to X$).

Let us consider the morphism

$$\phi = (\pi_A \circ (\xi \otimes \xi) - \xi \circ \pi_B) \ : \ q^*\mathcal{A} \otimes q^*\mathcal{A} \longrightarrow q^*\mathcal{B}$$

where $\pi_A \colon q^*\mathcal{A} \otimes q^*\mathcal{A} \to q^*\mathcal{A}$ is the morphism induced by the product of the algebra $\mathcal{A}$ and analogously for π_B. It is clear that $\xi|_\lambda$ is a morphism of $\mathcal{O}_T$-algebras if and only if $\phi|_\lambda$ vanishes. Now the existence of $\mathbf{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{A}, \mathcal{B})$ follows from A.21.

□

B Space of Jets

Let X and Y be differentiable spaces. We shall give a construction of the space $\mathbf{J}^r(X,Y)$ of r-jets of morphisms $X \to Y$. The language of differentiable spaces let us define this object in terms of its functor of points, i.e., $\mathbf{J}^r(X,Y)$ is a differentiable space which represents a certain functor. Typically, any relevant fibre bundle used in differential geometry is associated to some representable functor, and this fact must be considered a basic ingredient in the definition of such objets.

In our construction of $\mathbf{J}^r(X,Y)$, the concept of an infinitesimal neighbourhood has a main role. For example, if $\varphi\colon X \to Y$ is a morphism of differentiable spaces, then the r-jet $j^r_x\varphi$ of φ at a point $x \in X$ is defined to be the restriction of φ to the infinitesimal neighbourhood U^r_x of order r of x, that is to say, $j^r_x\varphi$ is the composition morphism $U^r_x \to X \xrightarrow{\varphi} Y$. The notion of an infinitesimal neighbourhood may be extend to parametrized points $x\colon T \to X$ in a natural way and, with this terminology, the space $\mathbf{J}^r(X,Y)$ is defined to be the differentiable X-space which represents the functor of r-jets,

$$F(T) := \operatorname{Hom}(U^r_x, Y)$$

for each parametrized point $x\colon T \to X$. Moreover, any morphism $\varphi\colon X \to Y$ defines a differentiable section of $\mathbf{J}^r(X,Y) \to X$,

$$j^r\varphi :\ X \longrightarrow \mathbf{J}^r(X,Y) \qquad , \qquad x \mapsto j^r_x\varphi$$

which is called the r-jet extension of φ.

We shall prove the existence of $\mathbf{J}^r(X,Y)$ when X is formally smooth. Moreover, we shall also show that $\mathbf{J}^r(X,Y)$ is endowed with a certain canonical form. This structure form is used to characterize when a differentiable section of $\mathbf{J}^r(X,Y) \to X$ is the r-jet extension of some morphism $X \to Y$.

B.1 Module of r-jets

Let A be an $\mathbb{R}$-algebra. We shall always consider on $A \otimes_{\mathbb{R}} A$ the structure of A-algebra induced by the morphism $A \to A \otimes_{\mathbb{R}} A$, $a \mapsto 1 \otimes a$. Recall that Δ_A denotes the kernel the morphism of algebras $A \otimes_{\mathbb{R}} A \dot{\to} A$, $(b,a) \mapsto ba$. The ideal Δ_A is generated by the increments $\Delta a = a \otimes 1 - 1 \otimes a$, $a \in A$.

Definition. The A-algebra of **algebraic r-jets** of A is defined to be

$$\mathfrak{J}^r_A := (A \otimes_{\mathbb{R}} A)/\Delta_A^{r+1} \ .$$

For any $a \in A$, the element $j^r a := [a \otimes 1] \in \mathfrak{J}^r_A$ is said to be the **r-jet** of a. Note that $\mathfrak{J}^r_A$ is generated by the elements $j^r a$ as an A-module.

Definition. Let M be an A-module. A differential operator $D\colon A \to M$ of order 0 is a morphism of A-modules. A $\mathbb{R}$-linear map $D\colon A \to M$ is said to be a **differential operator** of order $\leq r$ if the map

$$[D,a]\colon A \longrightarrow M \qquad , \qquad b \mapsto D(ab) - aDb$$

is a differential operator of order $\leq r-1$ for any $a \in A$.

Example. Let $\varphi\colon \mathbb{R}[x_1,\ldots,x_n] \to A$ be a morphism of $\mathbb{R}$-algebras. For each multi-index $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, let us define the following differential operator

$$\frac{\partial}{\partial x^\alpha}\colon \mathbb{R}[x_1,\ldots,x_n] \longrightarrow A \qquad , \qquad \frac{\partial}{\partial x^\alpha}(f) := \varphi\left(\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}\right)$$

where $|\alpha| = \sum \alpha_i$.

It easy to prove, by induction on r, that the set $\{\frac{\partial}{\partial x^\alpha}\}_{0\leq|\alpha|\leq r}$ is a basis of the A-module of all differential operators $D\colon \mathbb{R}[x_1,\ldots,x_n] \to A$ of order $\leq r$.

Proposition B.1. *Let A be an $\mathbb{R}$-algebra. We have:*

(a) *The map $j^r\colon A \to \mathfrak{J}^r_A$, $a \mapsto j^r a = [a \otimes 1]$, is a differential operator of order $\leq r$.*

(b) *Given a differential operator $D\colon A \to M$ of order $\leq r$, there exists a unique morphism of A-modules $\varphi\colon \mathfrak{J}^r_A \to M$ such that $D = \varphi \circ j^r$.*

Proof. (a) By induction on r. For $r = 0$ the result is clear, since

$$j^0\colon A \longrightarrow \mathfrak{J}^0_A = (A \otimes_{\mathbb{R}} A)/\Delta_A = A$$

is the identity map. For the general case, it is easy to check that the operator $[j^r, a]$ coincides with the composition map

$$A \xrightarrow{j^{r-1}} \mathfrak{J}^{r-1}_A \xrightarrow{\cdot(a\otimes 1 - 1\otimes a)} \mathfrak{J}^r_A$$

where the first map is a differential operator of order $\leq r-1$ (by the induction hypothesis) and the second one is a morphism of A-modules. Hence $[j^r, a]$ is a differential operator of order $\leq r-1$.

(b) It is easy to check that the A-linear map $\tilde{D}\colon A \otimes_{\mathbb{R}} A \to M$, $\tilde{D}(a \otimes b) := b \cdot Da$ is a differential operator of order $\leq r$ (where M is considered as an $A \otimes_{\mathbb{R}} A$-module via the product $(a \otimes b) \cdot m = abm$). Now, it is an elementary fact that if $D\colon A \to M$ is a differential operator of order $\leq r$ and I is an ideal of A, then $D(I^n) \subseteq I^{n-r}M$ for any $n \geq r$. In particular, we have $\tilde{D}(\Delta_A^{r+1}) \subseteq \Delta_A M = 0$, hence $\tilde{D}$ induces an A-linear map $\varphi\colon \mathfrak{J}^r_A \to M$, which has the desired property: $D = \varphi \circ j^r$.

Finally, the uniqueness of φ is clear, since $\mathfrak{J}^r_A$ is generated by the elements $j^r a$.

□

Let A be a differentiable algebra. Recall that the diagonal ideal $\mathcal{D}_A$ of $A\widehat{\otimes}_{\mathbb{R}}A$ is the kernel of the morphism $A\widehat{\otimes}_{\mathbb{R}}A \dot{\to} A$.

Definition. Let A be a differentiable algebra. The differentiable **A-algebra of r-jets** of A is defined to be

$$J^r_A \ := \ (A\widehat{\otimes}_{\mathbb{R}}A)/\,\overline{\mathcal{D}^{r+1}_A} \ .$$

For any $a \in A$, the element $j^r a := [a \otimes 1] \in J^r_A$ is said to be the **r-jet** of a. The map $j^r\colon A \to J^r_A$, $a \mapsto j^r a$, is a continuous differential operator of order $\leq r$.

Proposition B.2. *For any differentiable algebra A, we have:*

$$J^r_A \ = \ \widehat{\mathfrak{J}^r_A} \ .$$

Proof. By 6.1.b, the cokernel of locally convex A-modules

$$\Delta^{r+1}_A \ \longrightarrow \ A \otimes_{\mathbb{R}} A \ \longrightarrow \mathfrak{J}^r_A \ \longrightarrow \ 0$$

induces the cokernel of Fréchet A-modules

$$\widehat{\Delta^{r+1}_A} \ \longrightarrow \ A\widehat{\otimes}_{\mathbb{R}}A \ \longrightarrow \ \widehat{\mathfrak{J}}^r_A \ \longrightarrow 0 \ .$$

Each power I^n of any ideal I is dense in $\left(\overline{I}\right)^n$. In particular, Δ^{r+1}_A is dense in $\mathcal{D}^{r+1}_A$ (recall that Δ_A is dense in $\mathcal{D}_A$ by 10.2), and we obtain a topological isomorphism

$$(A\widehat{\otimes}_{\mathbb{R}}A)/\,\overline{\mathcal{D}^{r+1}_A} \ = \ \widehat{\mathfrak{J}}^r_A \ .$$

□

Theorem B.3. *Let A be a differentiable algebra and let M be a Fréchet A-module. Given a continuous differential operator $D\colon A \to M$ of order $\leq r$, there exists a unique morphism of Fréchet A-modules $\varphi\colon J^r_A \to M$ such that $D = \varphi \circ j^r$. That is to say,*

$$\mathrm{Diff}^r_{\mathbb{R}}(A, M) \ = \mathrm{Hom}_A(J^r_A, M) \ .$$

Proof. It is a direct consequence of B.1 and B.2.

□

Remark B.4. If A is a differentiable algebra, then any differential operator $D\colon A \to A$ of order $\leq r$ is continuous.

Proof. Since $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ has the quotient topology, it is enough to prove that any differential operator $D\colon \mathcal{C}^\infty(\mathbb{R}^n) \to A$ is continuous. Let

$$\overline{D} = \sum_{0\leq|\alpha|\leq r} a_\alpha \frac{\partial}{\partial^\alpha x} \qquad (a_\alpha \in A)$$

be the restriction of D to $\mathbb{R}[x_1, \dots, x_n]$. The same formula defines a continuous differential operator $\overline{D}\colon \mathcal{C}^\infty(\mathbb{R}^n) \to A$ which coincides with D on the algebra $\mathbb{R}[x_1, \dots, x_n]$. To prove that $D = \overline{D}$, let us show that if a differential operator $\tilde{D}\colon \mathcal{C}^\infty(\mathbb{R}^n) \to A$ of order $\leq r$ vanishes on $\mathbb{R}[x_1, \dots, x_n]$, then $\tilde{D} = 0$. For any point $p \in \mathrm{Spec}_r A \subseteq \mathbb{R}^n$, let us denote by $\mathfrak{m}_p$ and $\overline{\mathfrak{m}}_p$ the respective maximal ideals of $\mathcal{C}^\infty(\mathbb{R}^n)$ and A. For any $m \geq r$, we have that $\tilde{D}(\mathfrak{m}_p^m) \subseteq {\overline{\mathfrak{m}}_p}^{m-r}$, hence the map $\tilde{D}\colon \mathcal{C}^\infty(\mathbb{R}^n) \to A/{\overline{\mathfrak{m}}_p}^{m-r}$ factors through $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{m}_p^m$. Since any element of $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{m}_p^m$ is the equivalence class of some polynomial, it results that the map $\tilde{D}\colon \mathcal{C}^\infty(\mathbb{R}^n) \to A/{\overline{\mathfrak{m}}_p}^{m-r}$ vanishes. Now 2.17 let us conclude that $\tilde{D}\colon \mathcal{C}^\infty(\mathbb{R}^n) \to A$ vanishes.

□

Proposition B.5. *Let A be a differentiable algebra and let U be an open set in* $\mathrm{Spec}_r A$*. We have a topological isomorphism*

$$J^r_{A_U} \;=\; (J^r_A)_U \;.$$

Proof. The restriction morphism $A \to A_U$ induces a morphism of locally convex A-modules $\mathfrak{J}^r_A \to \mathfrak{J}^r_{A_U}$ and, by completion, we obtain a morphism $J^r_A \to J^r_{A_U}$. By the universal property of the localization, we obtain a morphism of locally convex A_U-modules

$$(J^r_A)_U \;\longrightarrow\; J^r_{A_U} \;.$$

Let us define the inverse morphism. If an element $s \in A$ does not vanish on U, then $1 \otimes s$ is invertible in the A_U-algebra $(J^r_A)_U$. Since $(s\otimes 1 - 1 \otimes s)^{r+1} = 0$ in $(J^r_A)_U$, we deduce that $s \otimes 1$ also is invertible in $(J^r_A)_U$. Now, the $\mathbb{R}$-linear map

$$D_r\colon A_U \;\longrightarrow\; (J^r_A)_U \quad , \quad D_r\Big(\frac{a}{s}\Big) = \frac{a\otimes 1}{s \otimes 1} \;,$$

is a differential operator of order $\leq r$ (by the same inductive argument used in the proof of B.1.a. Moreover, it is continuous because, by 8.7, the topology of A_U is the final topology of the map $A \times S \to A_U$, $(a, s) \mapsto a/s$. Hence D_r induces a continuous morphism of A_U-modules

$$J^r_{A_U} \;\longrightarrow\; (J^r_A)_U$$

which is just the inverse morphism.

□

Definition. Let $p\colon A \to B$ be a morphism of differentiable algebras. The kernel of the morphism $A\widehat{\otimes}_{\mathbb{R}} B \to B$, $a \otimes b \mapsto p(a)b$ will be denoted by $\mathcal{D}_p$. The **B-algebra of r-jets** of A at the (parametrized) point p is defined to be

$$J_p^r(A) := (A\widehat{\otimes}_{\mathbb{R}}B)/\overline{\mathcal{D}_p^{r+1}} .$$

When p is the identity map $id\colon A \to A$, by definition $J_{id}^r(A) = J_A^r$.

Let Δ_p be the kernel of the morphism $A\otimes_{\mathbb{R}} B \to B$, $a\otimes b \mapsto p(a)b$. The same arguments used in 10.2 and B.2 show that

$$\mathcal{D}_p = \widehat{\Delta}_p \quad , \quad J_p^r(A) = [(A\otimes_{\mathbb{R}} B)/\Delta_p^{r+1}]^\wedge .$$

Example B.6. Let A be a differentiable algebra and let $x \in \operatorname{Spec}_r A$. The $\mathbb{R}$-algebra of r-jets $J_x^r(A)$ at the point $x\colon A \to A/\mathfrak{m}_x$ is

$$J_x^r(A) = A/\mathfrak{m}_x^{r+1}$$

because the kernel of the morphism $(x, Id)\colon A\widehat{\otimes}_{\mathbb{R}}(A/\mathfrak{m}_x) = A \to A/\mathfrak{m}_x$ is just $\mathfrak{m}_x$, and $\mathfrak{m}_x^{r+1}$ is closed in A.

Proposition B.7. *Let $p\colon A \to B$ be a morphism of differentiable algebras. We have a topological isomorphism*

$$J_A^r\widehat{\otimes}_A B = J_p^r(A) .$$

Proof. The following exact sequence of A-modules

$$0 \longrightarrow \Delta \longrightarrow A\otimes_{\mathbb{R}} A \xrightarrow{(Id,Id)} A \longrightarrow 0$$

splits topologically, since we have the continuous section $A \to A\otimes_{\mathbb{R}} A$, $a \mapsto 1\otimes a$. Applying the functor $\otimes_A B$ we obtain that the exact sequence

$$0 \longrightarrow \Delta\otimes_A B \longrightarrow A\otimes_{\mathbb{R}} B \xrightarrow{(p,Id)} B \longrightarrow 0$$

splits topologically. In particular, we obtain a topological isomorphism

$$\Delta\otimes_A B = \Delta_p .$$

Now let us consider the following exact sequence of locally convex A-modules

$$0 \longrightarrow \Delta^{r+1} \longrightarrow A\otimes_{\mathbb{R}} A \xrightarrow{\pi} \mathfrak{J}_A^r \longrightarrow 0 ,$$

where π is an open map. Applying the functor $\otimes_A B$ we obtain an exact sequence

$$\Delta^{r+1}\otimes_A B \longrightarrow A\otimes_{\mathbb{R}} B \xrightarrow{\pi\otimes 1} \mathfrak{J}_A^r\otimes_A B \longrightarrow 0 ,$$

where $\pi\otimes 1$ is an open map (6.3.a). Taking generators, it is easy to check that the image of $\Delta^{r+1}\otimes_A B$ in $A\otimes_{\mathbb{R}} B$ is Δ_p^{r+1}. Therefore, we have a topological isomorphism

$$(A\otimes_{\mathbb{R}} B)/\Delta_p^{r+1} = \mathfrak{J}_A^r\otimes_A B$$

and by completion we conclude that

$$(A\widehat{\otimes}_{\mathbb{R}}B/\overline{\mathcal{D}_p^{r+1}} = J_A^r\widehat{\otimes}_A B .$$

□

By 10.10, we know that the diagonal ideal $\mathcal{D}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ of

$$\mathcal{C}^\infty(\mathbb{R}^n)\widehat{\otimes}_{\mathbb{R}}\mathcal{C}^\infty(\mathbb{R}^n) \;=\; \mathcal{C}^\infty(\mathbb{R}^{2n})$$

is generated by the increments

$$\Delta x_1 = x_1 \otimes 1 - 1 \otimes x_1 = x_1 - y_1, \ldots\ldots, \Delta x_n = x_n \otimes 1 - 1 \otimes x_n = x_n - y_n \ .$$

For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, let us denote

$$(\Delta x)^\alpha := (\Delta x_1)^{\alpha_1} \cdots (\Delta x_n)^{\alpha_n}$$

and $|\alpha| := \sum \alpha_i$.

Proposition B.8. *There exists an isomorphism of* $\mathcal{C}^\infty(\mathbb{R}^n)$*-algebras*

$$\mathcal{C}^\infty(\mathbb{R}^n)[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^{r+1} \;=\; J^r_{\mathcal{C}^\infty(\mathbb{R}^n)} \qquad , \qquad x_i \mapsto \Delta x_i \ .$$

Equivalently, $J^r_{\mathcal{C}^\infty(\mathbb{R}^n)}$ *is a free* $\mathcal{C}^\infty(\mathbb{R}^n)$*-module and* $\{(\Delta x)^\alpha\}_{0\le|\alpha|\le r}$ *is a basis.*

Proof. $\mathcal{D}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ is the ideal of all differentiable functions vanishing on the diagonal submanifold $x_1 = y_1, \ldots, x_n = y_n$ of $\mathbb{R}^{2n}$. By 2.7, we obtain that the increments $\Delta x_i = x_i - y_i$ generate $\mathcal{D}_{\mathcal{C}^\infty(\mathbb{R}^n)}$.

Given a differentiable function $f(x, y)$ on $\mathbb{R}^{2n}$, let $h_0(y) := f(y, y)$. Since $f(x, y) - h_0(y)$ vanishes on the diagonal submanifold, we have

$$f(x, y) = h_0(y) + \sum f_i(x, y)\Delta x_i$$

for some differentiable functions $f_i(x, y)$. Applying the above formula to the functions f_i, it results

$$f(x, y) = h_0(y) + \sum h_i(y)\Delta x_i + \sum f_{ij}(x, y)\Delta x_i \Delta x_j \ .$$

By recurrence, we may obtain that

$$(*) \qquad f(x, y) \equiv \sum_{0\le|\alpha|\le r} h_\alpha(y)(\Delta x)^\alpha \qquad \mathrm{mod}\, \mathcal{D}^{r+1}_{\mathcal{C}^\infty(\mathbb{R}^n)} \ .$$

Moreover, the above expression for $f(x, y)$ is unique, since by iterated partial derivation we have

$$h_\alpha(y) = D^\alpha f := \frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}(y, y) \ .$$

The formula $(*)$ and its uniqueness imply that $\mathcal{C}^\infty(\mathbb{R}^{2n})/\mathcal{D}^{r+1}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ is a free $\mathcal{C}^\infty(\mathbb{R}^n)$-module and $\{(\Delta x)^\alpha\}_{0\le|\alpha|\le r}$ is a basis. Moreover, $f \in \mathcal{D}^{r+1}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ if and only if $D^\alpha f = 0$ for any $0 \le |\alpha| \le r$, hence $\mathcal{D}^{r+1}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ is a closed ideal. In conclusion, we have that $J^r_{\mathcal{C}^\infty(\mathbb{R}^n)} = \mathcal{C}^\infty(\mathbb{R}^{2n})/\mathcal{D}^{r+1}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ is a free $\mathcal{C}^\infty(\mathbb{R}^n)$-module and $\{(\Delta x)^\alpha\}_{0\le|\alpha|\le r}$ is a basis.

□

Corollary B.9. *If $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ is a differentiable algebra, then the A-module J^r_A is finitely generated.*

Proof. The continuous epimorphism of $\mathbb{R}$-algebras

$$\mathcal{C}^\infty(\mathbb{R}^n)\widehat{\otimes}_{\mathbb{R}}\mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow A\widehat{\otimes}_{\mathbb{R}}A$$

clearly maps $\mathcal{D}_{\mathcal{C}^\infty(\mathbb{R}^n)}$ onto $\mathcal{D}_A$, hence it maps $\overline{\mathcal{D}^{r+1}_{\mathcal{C}^\infty(\mathbb{R}^n)}}$ onto $\overline{\mathcal{D}^{r+1}_A}$. Therefore, we obtain an epimorphism $J^r_{\mathcal{C}^\infty(\mathbb{R}^n)} \longrightarrow J^r_A$ and, by B.8, we conclude that J^r_A is a finitely generated A-module.

□

Corollary B.10. *Let A be a differentiable algebra and let $x \in \operatorname{Spec}_r A$. Then*

$$J^r_A \otimes_A (A/\mathfrak{m}_x) \;=\; A/\mathfrak{m}_x^{r+1} \;.$$

Proof. By B.6 and B.7 we have

$$A/\mathfrak{m}_x^{r+1} \;=\; J^r_x(A) \;=\; J^r_A\widehat{\otimes}_A(A/\mathfrak{m}_x) \;=\; J^r_A/\,\overline{\mathfrak{m}_x J^r_A} \;.$$

Since J^r_A is a finitely generated A-module, there exists a continuous epimorphism $A^m \to J^r_A$, which is an open map since both modules are Fréchet. Then

$$\mathbb{R}^m = A^m \otimes_A (A/\mathfrak{m}_x) \longrightarrow J^r_A \otimes_A (A/\mathfrak{m}_x) = J^r_A/\mathfrak{m}_x J^r_A$$

is an open epimorphism, hence $J^r_A/\mathfrak{m}_x J^r_A$ is separated, i.e., $\mathfrak{m}_x J^r_A = \overline{\mathfrak{m}_x J^r_A}$. We conclude that

$$J^r_A/\,\overline{\mathfrak{m}_x J^r_A} \;=\; J^r_A/\mathfrak{m}_x J^r_A \;=\; J^r_A \otimes_A (A/\mathfrak{m}_x) \;.$$

□

Proposition B.11. *Let $A \to A/\mathfrak{a} = B$ be an epimorphism of differentiable algebras. Then we have a cokernel of Fréchet B-modules*

$$\mathfrak{a}\widehat{\otimes}_{\mathbb{R}}B \xrightarrow{\;j^r\otimes 1\;} J^r_A\widehat{\otimes}_A B \longrightarrow J^r_B \longrightarrow 0 \;.$$

Proof. The desired sequence is obtained applying completion in the following cokernel of locally convex B-modules

$$\mathfrak{a} \otimes_k B \xrightarrow{\;j^r\otimes 1\;} \mathfrak{J}^r_A \otimes_A B \longrightarrow \mathfrak{J}^r_B \longrightarrow 0 \;.$$

□

Theorem B.12. *Let $B = \mathcal{C}^\infty(\mathbb{R}^n)/\,\boldsymbol{W}_Y$ be the Whitney algebra of a closed set Y in $\mathbb{R}^n$. There exists an isomorphism of B-algebras*

$$B[x_1,\dots,x_n]/(x_1,\dots,x_n)^{r+1} \;=\; J^r_B \qquad , \qquad x_i \mapsto \Delta x_i \;.$$

Equivalently, J^r_B is a free B-module and $\{(\Delta x)^\alpha\}_{0\le|\alpha|\le r}$ is a basis.

Proof. With the notations of the proof of B.8, for any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ we have

$$(j^r f) \otimes 1 = \sum_{0 \leq |\alpha| \leq r} [D^\alpha f] \cdot (\Delta x)^\alpha \in J^r_{\mathcal{C}^\infty(\mathbb{R}^n)} \widehat{\otimes}_{\mathcal{C}^\infty(\mathbb{R}^n)} B \ ,$$

where $[D^\alpha f]$ is the class in $B = \mathcal{C}^\infty(\mathbb{R}^n)/\mathbf{W}_Y$ of

$$D^\alpha f = \frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \ .$$

By definition of the Whitney ideal, we have $[D^\alpha f] = 0$ for any $f \in \mathbf{W}_Y$, hence $(j^r f) \otimes 1 = 0$ in $J^r_{\mathcal{C}^\infty(\mathbb{R}^n)} \widehat{\otimes}_{\mathcal{C}^\infty(\mathbb{R}^n)} B$. Applying the sequence B.11 to the case $A = \mathcal{C}^\infty(\mathbb{R}^n)$ and $\mathfrak{a} = \mathbf{W}_Y$, we obtain that

$$J^r_B \ = \ J^r_{\mathcal{C}^\infty(\mathbb{R}^n)} \widehat{\otimes}_{\mathcal{C}^\infty(\mathbb{R}^n)} B$$

and the result follows from B.8.

□

Definition. Let X be a differentiable space and let $\mathcal{D}_X$ be the sheaf of ideals of the diagonal embedding $X \hookrightarrow X \times X$ (rigorously, it is a sheaf of closed ideals on an open neighbourhood of the diagonal subspace). We say that the $\mathcal{O}_X$-algebra (with the structure induced by the second projection $p_2 \colon X \times X \to X$)

$$J^r_X := \mathcal{O}_{X \times X} / \overline{\mathcal{D}^{r+1}_X}$$

is the **sheaf of r-jets** of X.

For any affine open subset $U = \mathrm{Spec}_r A \subseteq X$, we have

$$J^r_X(U) \ = \ J^r_A \ .$$

According to B.8 and its corollaries, we have the following facts:

1. $J^r_{\mathbb{R}^n}$ is a free $\mathcal{O}_{\mathbb{R}^n}$-module. Moreover, if X is a formally smooth differentiable space, then J^r_X is a locally free $\mathcal{O}_X$-module (by 10.20 and B.12).
2. J^r_X is a locally finitely generated $\mathcal{O}_X$-module.
3. If $x \in X$, then $J^r_X \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathfrak{m}_x) \ = \ \mathcal{O}_X/\mathfrak{m}_x^{r+1}$.

Definition. Let X be a differentiable space and let $x \colon T \to X$ be a parametrized point of X. Let $\mathcal{D}_x$ be the sheaf of ideals defined by the graph $(x, Id) \colon T \hookrightarrow X \times T$ of x (rigorously, it is a sheaf of closed ideals on an open neighbourhood of the graph). We say that the $\mathcal{O}_T$-algebra (with the structure induced by the second projection $p_2 \colon X \times T \to T$)

$$J^r_x X := \ \mathcal{O}_{X \times T} / \overline{\mathcal{D}^{r+1}_x}$$

is the **sheaf of r-jets** of X at the parametrized point x.

According to B.7, we have

$$J^r_x X \ = \ J^r_X \widehat{\otimes}_{\mathcal{O}_X} \mathcal{O}_T \ = \ x^* J^r_X \ .$$

B.2 Jets of Morphisms

Let X be a differentiable space. Given a point $x \in X$, recall that the infinitesimal neighbourhood of order r of x in X is defined to be $U_x^r = \operatorname{Spec}_r(\mathcal{O}_{X,x}/\mathfrak{m}_x^{r+1})$. Now, we shall extend this notion to parametrized points.

Let $x\colon T \to X$ be a parametrized point of X. Recall that we denote by $\mathcal{D}_x$ the sheaf of ideals of the graph $T \hookrightarrow X \times T$ of x.

Definition. The **infinitesimal neighbourhood** of order r of the parametrized point x is defined to be the differentiable subspace U_x^r of $X \times T$ determined by the sheaf of ideals $\overline{\mathcal{D}_x^{r+1}}$, that is to say,

$$U_x^r = \operatorname{Spec}_r\left(\mathcal{O}_{X\times T}/\overline{\mathcal{D}_x^{r+1}}\right) = \operatorname{Spec}_r(J_x^r X) = \operatorname{Spec}_r(J_X^r \widehat{\otimes}_{\mathcal{O}_X} \mathcal{O}_T) .$$

The morphism $(x, Id)\colon T \hookrightarrow U_x^r$, which is a closed embedding and is called the **inclusion morphism**, admits the structural projection $U_x^r \to T$ as a retraction. On the other hand, we have a commutative triangle

$$\begin{array}{ccc} T & \xrightarrow{x} & X \\ & {\scriptstyle (x,Id)}\searrow \quad \nearrow{\scriptstyle p_1} & \\ & U_x^r & \end{array}$$

where $p_1\colon U_x^r \subseteq X \times T \to X$ is the first projection: *Any point factors through the inclusion into the infinitesimal neighbourhood.*

Let $f\colon X \to Y$ be a morphism of differentiable spaces, and let $y = f(x) = f \circ x\colon T \to Y$. The morphism $f \times Id\colon X \times T \to Y \times T$ transforms the graph of x into the graph of y, hence it induces a morphism $f \times Id\colon U_x^r \to U_y^r$ and we have a commutative diagram:

$$\begin{array}{ccc} U_x^r & \xrightarrow{f\times Id} & U_y^r \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Definition. Let X and Y be differentiable spaces. Let us consider the following contravariant functor on the category of differentiable X-spaces:

$$\mathcal{J}^r_{(X,Y)}(T) := \operatorname{Hom}(U_x^r, Y)$$

for each parametrized point $x\colon T \to X$. When this functor is representable by a differentiable space $\mathbf{J}^r(X,Y) \to X$, we say that $\mathbf{J}^r(X,Y)$ is the **space of jets** of order r of morphisms $X \to Y$.

By definition, for any parametrized point $x\colon T \to X$, we have a bijection

$$\operatorname{Hom}_X(T, \mathbf{J}^r(X,Y)) = \operatorname{Hom}(U_x^r, Y) .$$

In particular, *points of* $\mathbf{J}^r(X,Y)$ *over a point* $x \in X$ *correspond with morphisms of the* r*-th infinitesimal neighbourhood of* x *into* Y*.*

Note that we have a canonical X-morphism

$$q = (q_1, q_2)\colon \mathbf{J}^r(X,Y) \longrightarrow X \times Y ,$$

defined by the morphism of functors

$$\mathcal{J}^r_{(X,Y)}(T) = \operatorname{Hom}(U^r_x, Y) \longrightarrow \operatorname{Hom}(T,Y) = \operatorname{Hom}_X(T, X\times Y)$$

transforming each morphism $U^r_x \xrightarrow{f} Y$ into the composition $T \hookrightarrow U^r_x \xrightarrow{f} Y$.

The following result determines the functor of points of $\mathbf{J}^r(X,Y)$ on the category of differentiable $X \times Y$-spaces.

Fundamental lemma. *For any parametrized point* $(x,y)\colon T \to X \times Y$ *we have a bijection*

$$\operatorname{Hom}_{X\times Y}(T, \mathbf{J}^r(X,Y)) \;=\; \operatorname{Hom}_{\mathcal{O}_T\text{-alg}}(J^r_y Y, J^r_x X) \;=\; \operatorname{Hom}_{\mathcal{O}_T\text{-alg}}(y^* J^r_Y, x^* J^r_X) \; .$$

Proof. The second equality is a consequence of B.7. Let us prove the first one: To give a morphism $T \to \mathbf{J}^r(X,Y)$ over $X \times Y$ is just to give a morphism $f\colon U^r_x \to Y$ transforming x into y; that is to say, such that the following diagram is commutative:

$$\begin{array}{ccc} U^r_x & \xrightarrow{\;f\;} & Y \\ {\scriptstyle (x,Id)}\nwarrow & & \nearrow{\scriptstyle y} \\ & T & \end{array}$$

and to give such morphism f is just to give a T-morphism transforming the graph of x into the graph of y:

$$\begin{array}{ccc} U^r_x & \xrightarrow{\;(f,Id)\;} & Y \times T \\ {\scriptstyle (x,Id)}\nwarrow & & \nearrow{\scriptstyle (y,Id)} \\ & T & \end{array}$$

or, equivalently, to give a commutative diagram of T-morphisms

$$\begin{array}{ccc} U^r_x & \xrightarrow{\;(f,Id)\;} & U^r_y \\ {\scriptstyle (x,Id)}\nwarrow & & \nearrow{\scriptstyle (y,Id)} \\ & T & \end{array}$$

We conclude since $U^r_x = \operatorname{Spec}_r(J^r_x X)$ and $U^r_y = \operatorname{Spec}_r(J^r_y Y)$.

□

Let us consider the following contravariant functor on the category of differentiable $X \times Y$-spaces:

$$F(T) \;=\; \operatorname{Hom}_{\mathcal{O}_T\text{-alg}}(y^* J^r_Y, x^* J^r_X) \;=\; \operatorname{Hom}_{\mathcal{O}_T\text{-alg}}(J^r_Y|_y, J^r_X|_x)$$

for any $X\times Y$-space $(x,y)\colon T\to X\times Y$. The fundamental lemma states that the functor F is representable by the space of jets $\mathbf{J}^r(X,Y)$ (if it exists). In other words,

$$\mathbf{J}^r(X,Y) \;=\; \mathbf{Hom}_{\mathcal{O}_{X\times Y}\text{-alg}}(p_2^*J_Y^r, p_1^*J_X^r)\ .$$

Now, if X is formally smooth, then J_X^r is a locally free $\mathcal{O}_X$-module of finite rank and, according to A.22, there exists the differentiable space

$$\mathbf{Hom}_{\mathcal{O}_{X\times Y}\text{-alg}}(p_2^*J_Y^r, p_1^*J_X^r)\ .$$

In conclusion, we have proved the following result.

Theorem B.13. *If X is formally smooth, then the space of jets $\mathbf{J}^r(X,Y)$ exists.*

According to Yoneda's lemma 7.1, the isomorphism of functors

$$\mathbf{J}^r(X,Y)^\bullet \;=\; F$$

is determined by an element $\delta\in F(\mathbf{J}^r(X,Y))$, which is a morphism of $\mathcal{O}_{\mathrm{J}}$-algebras

$$\delta\ :\ q_2^*J_Y^r \longrightarrow q_1^*J_X^r$$

on $\mathbf{J}=\mathbf{J}^r(X,Y)$. It is called the **universal morphism** and it let us describe the given isomorphism of functors in the following way: For any $X\times Y$-space $(x,y)\colon T\to X\times Y$, we have a bijection

$$\begin{aligned}\mathrm{Hom}_{X\times Y}(T,\mathbf{J}^r(X,Y)) \;&=\; \mathrm{Hom}_{\mathcal{O}_T\text{-alg}}(J_Y^r|_y, J_X^r|_x)\\ \lambda \;&\mapsto\; \delta|_\lambda\end{aligned}$$

Theorem B.14. *If X and Y are formally smooth, then*

$$q=(q_1,q_2)\ :\ \mathbf{J}^r(X,Y)\longrightarrow X\times Y$$

is a smooth morphism. Therefore, if X and Y are smooth manifolds, then so is $\mathbf{J}^r(X,Y)$.

Proof. Since the problem is local, we may assume that X is a Whitney subspace of $\mathbb{R}^n$ and that Y is a Whitney subspace of $\mathbb{R}^m$ (see 10.20). According to B.12, we have isomorphisms of algebras

$$\begin{aligned}J_X^r \;&=\; \mathcal{O}_X[\Delta x_1,\dots,\Delta x_n]/(\Delta x_1,\dots,\Delta x_n)^{r+1}\\ J_Y^r \;&=\; \mathcal{O}_Y[\Delta y_1,\dots,\Delta y_m]/(\Delta y_1,\dots,\Delta y_m)^{r+1}\end{aligned}$$

Let $\mathcal{L}$ be the free $\mathcal{O}_Y$-submodule of J_Y^r generated by $\{\Delta y_1,\dots,\Delta y_m\}$. It is clear that

$$\mathbf{J}^r(X,Y) \;=\; \mathbf{Hom}_{\mathcal{O}_{X\times Y}\text{-alg}}(p_2^*J_Y^r, p_1^*J_X^r) \;=\; \mathbf{Hom}_{\mathcal{O}_{X\times Y}}(p_2^*\mathcal{L}, p_1^*J_X^r)\ .$$

Since $p_2^*\mathcal{L}=p_2^*(\mathcal{O}_Y^m)=\mathcal{O}_{X\times Y}^m$, we have an isomorphism

$$\mathbf{Hom}_{\mathcal{O}_{X\times Y}}(p_2^*\mathcal{L}, p_1^*J_X^r) \;=\; \mathbf{Hom}_{\mathcal{O}_{X\times Y}}(\mathcal{O}_{X\times Y}, (p_1^*J_X^r)^m)$$

hence $q\colon \mathbf{J}^r(X,Y)\to X\times Y$ is the trivial vector bundle associated to the free $\mathcal{O}_{X\times Y}$-module $(p_1^*J_X^r)^m$.

□

B.3 Tangent Bundle of order r

Definition. Let X be a differentiable space. The **cotangent module of order** r of X is defined to be the Fréchet $\mathcal{O}_X$-module

$$\Omega_X^r := \mathcal{D}_X / \overline{\mathcal{D}_X^{r+1}}$$

where $\mathcal{D}_X$ is the sheaf of ideals of the diagonal embedding $X \hookrightarrow X \times X$. The $\mathcal{O}_X$-module structure of Ω_X^r is induced by the second projection $p_2 \colon X \times X \to X$. The cotangent module of order 1 is just the $\mathcal{O}_X$-module of relative differentials Ω_X.

Note that we have an exact sequence

$$0 \longrightarrow \Omega_X^r \longrightarrow J_X^r = \mathcal{O}_{X \times X} / \overline{\mathcal{D}_X^{r+1}} \stackrel{\cdot}{\longrightarrow} \mathcal{O}_X \longrightarrow 0 ,$$

which splits because we have the section $\mathcal{O}_X \to J_X^r$, $a \mapsto [1 \otimes a]$, hence we obtain a canonical decomposition

$$J_X^r = \mathcal{O}_X \oplus \Omega_X^r .$$

Given a point $x \in X$, tensoring by $\mathcal{O}/\mathfrak{m}_x$ in the above exact sequence and using B.10, we obtain an isomorphism

$$\Omega_X^r \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_x = \mathfrak{m}_x/\mathfrak{m}_x^{r+1} .$$

Definition. Let X be a differentiable space and let $\mathcal{M}$ be a locally convex $\mathcal{O}_X$-module. A $\mathbb{R}$-linear morphism of sheaves $D \colon \mathcal{O}_X \to \mathcal{M}$ is said to be a **(continuous) differential operator of order** $\leq r$ if, for any open set $U \subseteq X$, the map $D \colon \mathcal{O}_X(U) \to \mathcal{M}(U)$ is a (resp. continuous) differential operator of order $\leq r$.

Differential operators $D \colon \mathcal{O}_X \to \mathcal{M}$ of order $\leq r$ such that $D(\mathbb{R}) = 0$ are named **derivations of order** $\leq r$.

Note that any differential operator $D \colon \mathcal{O}_X \to \mathcal{O}_X$ of order $\leq r$ is continuous by B.4.

We have a natural map

$$\mathrm{d}^r : \mathcal{O}_X \longrightarrow \Omega_X^r \quad , \quad \mathrm{d}^r(a) := a \otimes 1 - 1 \otimes a .$$

Note that the universal differential operator $j^r \colon \mathcal{O}_X \to J_X^r = \mathcal{O}_X \oplus \Omega_X^r$ is $j^r = Id \oplus \mathrm{d}^r$, since $j^r a = a \otimes 1 = 1 \otimes a + (a \otimes 1 - 1 \otimes a)$. From theorem B.3, we may easily deduce that for any continuous derivation $D \colon \mathcal{O}_X \to \mathcal{M}$ of order $\leq r$ ($\mathcal{M}$ being a Fréchet $\mathcal{O}_X$-module), there exists a unique morphism of Fréchet $\mathcal{O}_X$-modules $\varphi \colon \Omega_X^r \to \mathcal{M}$ such that $D = \varphi \circ \mathrm{d}^r$, that is to say,

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_X^r, \mathcal{M}) = \mathrm{Der}_{\mathbb{R}}^r(\mathcal{O}_X, \mathcal{M}) .$$

Definition. Let X be a differentiable space. The $\mathcal{O}_X$-module

$$\mathcal{T}_X^r := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^r, \mathcal{O}_X) \;=\; \mathcal{D}er_{\mathbb{R}}^r(\mathcal{O}_X, \mathcal{O}_X)$$

is said to be the **tangent module of order** r of X. For any open set $U \subseteq X$, we have

$$\mathcal{T}_X^r(U) := \mathrm{Der}_{\mathbb{R}}^r(\mathcal{O}_U, \mathcal{O}_U) \;=\; \{\text{derivations } D\colon \mathcal{O}_U \to \mathcal{O}_U \text{ of order } \leq r\} \; .$$

When $r = 1$, the tangent module is the sheaf of tangent vector fields on X.

Definition. Let X be formally smooth differentiable space. Then J_X^r is a locally free $\mathcal{O}_X$-module of finite rank, hence so are its direct summand Ω_X^r and $\mathcal{T}_X^r$. The **tangent bundle of order r** of X is defined to be the vector bundle $\mathbb{T}^r X \to X$ associated to the locally free $\mathcal{O}_X$-module $\mathcal{T}_X^r$.

Given a point $x \in X$, the fibre of the tangent bundle $\mathbb{T}^r X \to X$ over x is called the **tangent space of order** r of X at x, and it is denoted by $T_x^r X$. It is easy to check that

$$T_x^r X \;=\; \mathrm{Der}_{\mathbb{R}}^r(\mathcal{O}_x, \mathbb{R}) \;=\; \{\text{derivations } D\colon \mathcal{O}_x \to \mathcal{O}_x/\mathfrak{m}_x = \mathbb{R} \text{ of order } \leq r\} \; .$$

Let $\varphi\colon X \to Y$ be a morphism of differentiable spaces. The continuous differential operator

$$\mathcal{O}_Y \xrightarrow{\varphi^*} \varphi_*\mathcal{O}_X \xrightarrow{j^r} \varphi_* J_X^r$$

induces a morphism of Fréchet $\mathcal{O}_Y$-algebras

$$J_Y^r \longrightarrow \varphi_* J_X^r$$

or, equivalently, a morphism of Fréchet $\mathcal{O}_X$-algebras

$$\varphi^* J_Y^r \xrightarrow{\varphi^*} J_X^r \; .$$

By the representability property of $\mathbf{J}^r(X,Y)$, i.e., by the fundamental lemma, the above morphism φ^* corresponds with a morphism of $X \times Y$-spaces

$$X \xrightarrow{j^r\varphi} \mathbf{J}^r(X,Y) \; .$$

In other words, $j^r\varphi\colon X \to \mathbf{J}^r(X,Y)$ is the unique morphism of $X \times Y$-spaces such that

$$\delta|_{j^r\varphi} \;=\; \varphi^* \; ,$$

where $\delta\colon q_2^* J_Y^r \to q_1^* J_X^r$ is the universal morphism.

Definition. The morphism $j^r\varphi$ is said to be the r**-jet extension** of φ.

Note. Let us give an alternative definition of the r-jet extension of φ in terms of the functors of points. For any parametrized point $x\colon T \to X$, the r-jet of φ at x is defined to be the composition $j_x^r\varphi\colon U_x^r \to X \xrightarrow{\varphi} Y$. By definition of the space of jets, the morphism $j^r_x\varphi$ defines a parametrized point $j^r_x\varphi\colon T \to \mathbf{J}^r(X,Y)$. Now, the r-jet extension of φ is the morphism of X-spaces $j^r_x\varphi\colon X \to \mathbf{J}^r(X,Y)$ defined by the following morphism between the corresponding functors of points:

$$X^\bullet(T) \longrightarrow \mathbf{J}^r(X,Y)^\bullet(T) \qquad , \qquad x \mapsto j^r_x\varphi \ .$$

Let us go back to the morphism $\varphi\colon X \to Y$. Since $J^r_X = \mathcal{O}_X \oplus \Omega^r_X$, it is clear that the morphism of $\mathcal{O}_X$-algebras $\varphi^*\colon \varphi^* J^r_Y \to J^r_X$ induces a morphism of Fréchet $\mathcal{O}_X$-modules

$$\varphi^*\Omega^r_Y \xrightarrow{\varphi^*} \Omega^r_X \ .$$

Taking duals and assuming that Y is formally smooth (i.e., Ω^r_Y is a locally free $\mathcal{O}_Y$-module), we finally obtain a morphism of $\mathcal{O}_X$-modules

$$\mathcal{T}^r_X \xrightarrow{\mathrm{d}^r\varphi} \varphi^*\mathcal{T}^r_Y \ .$$

Definition. The morphism $\mathrm{d}^r\varphi$ is said to be the **differential of order** r of φ. In order to simplify the notation, sometimes $\mathrm{d}^r\varphi$ will be denoted by φ.

Note that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{T}^r X & \xrightarrow{\mathrm{d}^r\varphi} & \mathbb{T}^r Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{\varphi} & Y \end{array}$$

The morphism $\mathrm{d}^r\varphi\colon \mathbb{T}^r X \longrightarrow \mathbb{T}^r Y$ induces a morphism between the fibres: $\mathrm{d}^r_x\varphi\colon T^r_x X \to T^r_y Y$. It not difficult to check that this morphism $\mathrm{d}^r_x\varphi$ may be described in the following way,

$$T^r_x X = \mathrm{Der}^r_{\mathbb{R}}(\mathcal{O}_x, \mathbb{R}) \xrightarrow{\mathrm{d}^r_x\varphi} \mathrm{Der}^r_{\mathbb{R}}(\mathcal{O}_y, \mathbb{R}) = T_y Y \ , \quad [(\mathrm{d}^r_x\varphi)(D)](f) = D(f \circ \varphi) \ .$$

B.4 Structure Form

Let X and Y be formally smooth differentiable spaces. Let us consider the space of r-jets

$$q = (q_1, q_2)\colon \ \mathbf{J} = \mathbf{J}^r(X,Y) \longrightarrow X \times Y \ .$$

Definition. The universal morphism of $\mathcal{O}_{\mathbf{J}}$-algebras $\delta\colon q_2^* J^r_Y \longrightarrow q_1^* J^r_X$ induces, via the decomposition $J^r_X = \mathcal{O}_X \oplus \Omega^r_X$ and duality, a morphism of Fréchet $\mathcal{O}_{\mathbf{J}}$-modules

$$q_1^*\mathcal{T}^r_X \xrightarrow{\mathrm{d}^r} q_2^*\mathcal{T}^r_Y$$

called the **universal differential of order** r.

Given a morphism $\varphi\colon X \to Y$, recall that the r-jet extension is the unique morphism $j^r\varphi\colon X \to \mathbf{J}^r(X,Y)$ of $X \times Y$-spaces such that $\delta|_{j^r\varphi} = \varphi^*$. This condition is equivalent to the following one:

$$\boxed{\mathrm{d}^r|_{j^r\varphi} \;=\; \mathrm{d}^r\varphi}$$

Definition. The **structure form** on $\mathbf{J} = \mathbf{J}^r(X,Y)$ is defined to be the morphism of Fréchet $\mathcal{O}_{\mathbf{J}}$-modules

$$\theta \;:=\; (q_2 - \mathrm{d}^r \circ q_1) \;:\; \mathcal{T}^r_{\mathrm{J}} \;\longrightarrow\; q_2^*\mathcal{T}^r_Y \;.$$

The structure form θ encodes the non-commutativeness of the triangle

$$(*) \qquad \begin{array}{ccc} & \mathcal{T}^r_{\mathrm{J}} & \\ {\scriptstyle q_1}\swarrow & & \searrow{\scriptstyle q_2} \\ q_1^*\mathcal{T}^r_X & \xrightarrow{\;\mathrm{d}^r\;} & q_2^*\mathcal{T}^r_Y \end{array}$$

Notation. Given a differentiable section $F\colon X \to \mathbf{J}^r(X,Y)$ of q_1, we denote by $F^*(\theta)$ the composition

$$\mathcal{T}^r_X \xrightarrow{\;F\;} F^*\mathcal{T}^r_{\mathrm{J}} \xrightarrow{\;\theta|_F\;} F^*q_2^*\mathcal{T}^r_Y = f^*\mathcal{T}^r_Y$$

where $f = q_2 \circ F\colon\; X \to Y$.

Theorem B.15. *Let X and Y be formally smooth differentiable spaces. Let $F\colon X \to \mathbf{J}^r(X,Y)$ be a differentiable section of q_1 and let $f = q_2 \circ F$. Then we have:*

$$F^*(\theta) = 0 \;\Leftrightarrow\; F \;=\; j^r f \;.$$

Proof. Since $f = q_2 \circ F$, we have a commutative triangle

$$\begin{array}{ccc} & F^*\mathcal{T}^r_{\mathrm{J}} & \\ {\scriptstyle F}\nearrow & & \searrow{\scriptstyle q_2} \\ \mathcal{T}^r_X & \xrightarrow{\;\mathrm{d}^r f\;} & f^*\mathcal{T}^r_Y \end{array}$$

On the other hand, applying F^* to the triangle (∗), we obtain a diagram

$$\begin{array}{ccc} & F^*\mathcal{T}^r_{\mathrm{J}} & \\ {\scriptstyle F}\nearrow\swarrow{\scriptstyle q_1} & & \searrow{\scriptstyle q_2} \\ \mathcal{T}^r_X & \xrightarrow[\;\mathrm{d}^r|_F\;]{} & f^*\mathcal{T}^r_Y \end{array}$$

where $q_1 \circ F = Id$ by hypothesis. Now, $F^*(\theta) = 0$ if and only if this triangle is commutative since

$$F^*(\theta) \;=\; \theta|_F \circ F \;=\; (q_2 - (\mathrm{d}^r|_F) \circ q_1) \circ F \;=\; q_2 \circ F - \mathrm{d}^r|_F \;.$$

Comparing both triangles we conclude that

$$F^*(\theta) = 0 \;\Leftrightarrow\; \mathrm{d}^r|_F = \mathrm{d}^r f \;,$$

hence $F^*(\theta) = 0 \;\Leftrightarrow\; F \;=\; j^r f$.

□

B.5 Jets of Sections

Let $\pi\colon E \to X$ be a morphism of differentiable spaces.

Definition. The **space of jets** of order r of sections of π is defined to be the differentiable X-space $\mathbf{J}^r_{E/X}$, if it exists, such that we have an isomorphism of functors on the category of differentiable X-spaces:

$$\mathrm{Hom}_X(T, \mathbf{J}^r_{E/X}) \;=\; \mathrm{Hom}_X(U^r_x, E)$$

for any parametrized point $x\colon T \to X$. *Points of $\mathbf{J}^r_{E/X}$ over a point x of X correspond with sections of $E \to X$ over the r-th infinitesimal neighbourhood of x in X.*

Theorem B.16. *If X is formally smooth, then the space of jets $\mathbf{J}^r_{E/X}$ exists. Moreover, it admits a canonical closed embedding $\mathbf{J}^r_{E/X} \hookrightarrow \mathbf{J}^r(X, E)$.*

Proof. Let $\delta\colon q_2^* J^r_E \to q_1^* J^r_X$ be the universal morphism of algebras on $\mathbf{J}^r(X,E)$ and let $\pi\colon q_1^* J^r_X \to q_2^* J^r_E$ be the morphism induced by $\pi\colon E \to X$. $\mathbf{J}^r_{E/X}$ is the closed differentiable subspace of all points of $\mathbf{J}^r(X,E)$ where the morphism $\delta \circ \pi - Id$ vanishes. It exists by A.21.

□

Theorem B.17. *Let X be a formally smooth differentiable space. Let $E \to X$ be the associated vector bundle of a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank. Then $\mathbf{J}^r_{E/X} \to X$ is the associated vector bundle of $J^r_X \otimes_{\mathcal{O}_X} \mathcal{E}$.*

Proof. Given a parametrized point $x\colon T \to X$, let us consider the infinitesimal neighbourhood of order r

$$U^r_x \;=\; \mathrm{Spec}_r\left(\mathcal{O}_{X\times T}/\,\overline{\mathcal{D}^{r+1}_x}\right) \;=\; \mathrm{Spec}_r\,(J^r_x X) \;=\; \mathrm{Spec}_r\,(J^r_X \widehat{\otimes}_{\mathcal{O}_X} \mathcal{O}_T)\;,$$

the closed embedding $(x, Id)\colon T \hookrightarrow U^r_x$ and the natural morphism $p_1\colon U^r_x \to X$. We have the following bijections

$$\mathrm{Hom}_X(T, \mathbf{J}^r_{E/X}) = \mathrm{Hom}_X(U^r_x, E) = \Gamma(U^r_x, p_1^*\mathcal{E}) = \Gamma(U^r_x, \mathcal{E} \otimes_{\mathcal{O}_X} [J^r_X \widehat{\otimes}_{\mathcal{O}_X} \mathcal{O}_T])$$

(since $T \hookrightarrow U^r_x$ is a homeomorphism)

$$=\; \Gamma(T, [\mathcal{E} \otimes_{\mathcal{O}_X} J^r_X] \widehat{\otimes}_{\mathcal{O}_X} \mathcal{O}_T) \;=\; \Gamma(T, x^*[\mathcal{E} \otimes_{\mathcal{O}_X} J^r_X])\;,$$

hence $\mathbf{J}^r_{E/X}$ is the associated vector bundle of $\mathcal{E} \otimes_{\mathcal{O}_X} J^r_X$.

□

From now on, we shall assume that X is formally smooth and that $\pi\colon E \to X$ is a smooth morphism (hence E is formally smooth).

Let $\overline{\theta}$ and $\overline{\mathrm{d}}$ be the restriction to $\mathbf{J}^r_{E/X}$ of the structure form θ and the universal differential d^r on $\mathbf{J}^r(X,E)$ respectively. We have a <u>non</u>-commutative diagram (we write $\overline{\mathbf{J}} = \mathbf{J}^r_{E/X}$):

$$\begin{array}{ccc} & \mathcal{T}^r_{\overline{\mathrm{J}}} & \\ {}^{q_1}\swarrow & & \searrow^{q_2} \\ q_1^*\mathcal{T}^r_X & \xrightarrow{\overline{\mathrm{d}}} & q_2^*\mathcal{T}^r_E \end{array}$$

where $\pi \circ q_2 = q_1$, $\pi \circ \overline{\mathrm{d}} = Id$, and, by definition, $\overline{\theta} = q_2 - \overline{\mathrm{d}} \circ q_1$.

Definition. Since π is a smooth morphism, then $\pi\colon \mathcal{T}^r_E \to \pi^*\mathcal{T}^r_X$ is an epimorphism, and we obtain an exact sequence of locally free $\mathcal{O}_E$-modules of finite rank

$$0 \longrightarrow V^r_{E/X} \longrightarrow \mathcal{T}^r_E \xrightarrow{\pi} \pi^*\mathcal{T}^r_X \longrightarrow 0$$

where $V^r_{E/X} := \ker \pi$ is said to be the **vertical tangent module** of order r. Remark that $\overline{\theta}$ is valued in $q_2^* V^r_{E/X}$:

$$\pi \circ \overline{\theta} \ = \ \pi \circ q_2 - \pi \circ \overline{\mathrm{d}} \circ q_1 \ = \ q_1 - q_1 \ = \ 0 \,,$$

so that $\overline{\theta}$ is a morphism of $\mathcal{O}_{\overline{\mathrm{J}}}$-modules $\overline{\theta}\colon \mathcal{T}^r_{\overline{\mathrm{J}}} \to q_2^* V^r_{E/X}$.

Notation. If $F\colon X \to \mathbf{J}^r_{E/X}$ is a differentiable section of q_1, then we denote by $F^*(\overline{\theta}\,)$ the composition

$$\mathcal{T}^r_X \xrightarrow{F} F^*\mathcal{T}^r_{\overline{\mathrm{J}}} \xrightarrow{\overline{\theta}} f^* V^r_{E/X} \,,$$

where $f := q_2 \circ F\colon X \to E$.

Theorem B.18. *Let X be a formally smooth space and let $\pi\colon E \to X$ be a smooth morphism. If $F\colon X \to \mathbf{J}^r_{E/X}$ is a differentiable section of q_1, and we put $f = q_2 \circ F$, then we have:*

$$F^*(\overline{\theta}) = 0 \ \Leftrightarrow \ F \ = \ j^r f \ .$$

Proof. Just apply theorem B.15 to $\mathbf{J}^r(X, E)$.

□

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